4.1 (a) We are given $p = 2$, $\mu = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \frac{2}{3} & -0.8 \times \sqrt{2} \\ -0.8 \times \sqrt{2} & 1 \end{bmatrix}$ so $|\Sigma| = 0.72$ and 

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{0.72} & \frac{\sqrt{2}}{0.72} \\ \frac{\sqrt{2}}{0.72} & \frac{2}{0.72} \end{bmatrix}$$

$$f(x) = \frac{1}{(2\pi)^{0.72}} \exp \left( -\frac{1}{2} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} ^2 + \frac{2\sqrt{2}}{0.72} (x_1 - 1)(x_2 - 3) + \frac{2}{0.72} (x_2 - 3)^2 \right)$$

(b) 

$$\frac{1}{0.72} (x_1 - 1)^2 + \frac{2\sqrt{2}}{0.72} (x_1 - 1)(x_2 - 3) + \frac{2}{0.72} (x_2 - 3)^2$$

4.3 We apply Result 4.5 that relates zero covariance to statistical independence

a) No, $\sigma_{12} \neq 0$

b) Yes, $\sigma_{23} = 0$

c) Yes, $\sigma_{13} = \sigma_{23} = 0$

d) Yes, by Result 4.3, $(X_1 + X_2)/2$ and $X_3$ are jointly normal and their covariance is $\frac{1}{2} \sigma_{13} + \frac{1}{2} \sigma_{23} = 0$

e) No, by Result 4.3 with $A = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 1 & -1 \end{bmatrix}$. Form $A \cdot A'$ to see that the covariance is 10 and not 0.

4.4 a) $3X_1 - 2X_2 + X_3$ is $N(13, 9)$

b) Require $\text{Cov}(X_1X_2 - x_1X_3 - 2x_3^2) = 3 - a_1 - 2a_3 = 0$. Thus any $a' = [a_1, a_3]$ of the form $a' = [3-2a_3, a_3]$ will meet the requirement. As an example, $a' = [1, 1]$.

c) $X_1 \mid X_2$ is $N(\frac{1}{2}x_2 - 2, \frac{3}{2})$

4.5 a) $X_2 \mid X_1, X_3$ is $N(-x_1 - 5, 1)$

c) $X_3 \mid X_1, X_2$ is $N(\frac{1}{2}(x_1 + x_2 + 3), \frac{1}{2})$
4.6 (a) $X_1$ and $X_2$ are independent since they have a bivariate normal distribution with covariance $\sigma_{12} = 0$.

(b) $X_1$ and $X_3$ are dependent since they have nonzero covariance $\sigma_{13} = -1$.

(c) $X_2$ and $X_3$ are independent since they have a bivariate normal distribution with covariance $\sigma_{23} = 0$.

(d) $X_1, X_2$ and $X_3$ are independent since they have a trivariate normal distribution where $\sigma_{12} = 0$ and $\sigma_{23} = 0$.

(e) $X_1$ and $X_1 + 2X_2 - 3X_3$ are dependent since they have nonzero covariance

$$\sigma_{11} + 2\sigma_{12} - 3\sigma_{13} = 4 + 2(0) - 3(-1) = 7$$

4.7 (a) $X_1|X_2$ is $N(1 + .5(x_3 - 2), 3.5)$

(b) $X_2|X_3$ is $N(1 + .5(x_3 - 2), 3.5)$ Since $X_2$ is independent of $X_1$, conditioning further on $x_2$ does not change the answer from Part a).

4.15 First,

$$\sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})' = n\sum_{j=1}^{n} (x_j - \bar{x})^2 = 0$$

Also,

$$\sum_{j=1}^{n} (x_j - \bar{x})(x_j - \bar{x})' = n\sum_{j=1}^{n} (x_j - \bar{x})^2 = 0$$

4.16 (a) By Result 4.8, with $c_1 = c_3 = 1/4$, $c_2 = c_4 = -1/4$ and $\mu_j = \mu$ for $j = 1, \ldots, 4$ we have $\sum_{j=1}^{4} c_j \mu_j = 0$ and $(\sum_{j=1}^{4} c_j^2) \Sigma = \frac{1}{4} \Sigma$. Consequently, $V_1$ is $N(0, \frac{1}{4} \Sigma)$. Similarly, setting $b_1 = b_2 = 1/4$ and $b_3 = b_4 = -1/4$, we find that $V_2$ is $N(0, \frac{1}{4} \Sigma)$.

(b) Again by Result 4.8, we know that $V_1$ and $V_2$ are jointly multivariate normal with covariance

$$\sum_{j=1}^{4} b_j c_j = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \Sigma = 0$$

That is,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

is distributed $N_{2p} \left( \left[ \frac{1}{4} \Sigma \right], 0 \right)$

so the joint density of the $2p$ variables is

$$f(v_1, v_2) = \frac{1}{(2\pi)^{p} \frac{1}{4} \Sigma} \exp \left( -\frac{1}{2} \left[ v_1, v_2 \right] \left[ \frac{1}{4} \Sigma \right]^{-1} \left[ v_1 \right] \right)$$

$$= \frac{1}{(2\pi)^{p} \frac{1}{4} \Sigma} \exp \left( -\frac{1}{8} \left( v_1 \Sigma^{-1} v_1 + v_2 \Sigma^{-1} v_2 \right) \right)$$