Statistics 13 Elementary Statistics
Summer Session I 2012

Lecture Notes 3: Probability

Definition 3.1 An experiment is an act or process of observation that leads to a single outcome that cannot be predicted with certainty.

Definition 3.2 A sample point is the most basic outcome of an experiment.

Example 1 A coin is tossed, and their up face is recorded. All the sample points are $H, T$.

Example 2 Two coins are tossed, and their up faces are recorded. All the sample points are $HH, HT, TH, TT$.

Definition 3.3 The sample space, labeled $S$, of an experiment is the collection of all its sample points.

Example 3 In Example 1 and 2, the sample spaces are $\{H, T\}$ and $\{HH, HT, TH, TT\}$ respectively.

Probability Rules for Sample Points
Let $p_i$ represent the probability of sample point $i$. Then

1. All sample point probabilities must lie between 0 and 1 (i.e. $0 \leq p_i \leq 1$).

2. The probabilities of all the sample points within a sample space must sum to 1 (i.e., $\sum p_i = 1$)

Example 4 In example 1, we have only two sample points and thus, we can write the corresponding two probabilities $p_1$ and $p_2$ for $H$ and $T$. If the coin is fair, $p_1 = p_2$. Since, by definition, $p_1 + p_2 = 1$, we have $p_1 = p_2 = 0.5$.

Definition 3.4 An event is a specific collection of sample points.

Example 5 In Example 1, $\{H\}$ is an event, it corresponds to having a head. $\{H, T\}$ is another event, it corresponds to having either head or tail.

Probability of an Event
The probability of an event $A$ is calculated by summing the probabilities of the sample points relevant to $A$. 

---

1Last update: June 26, 2012
points in the sample space for $A$.

**Example 6** Refer to Example 4 and 5. If $A = \{H\}$, $P(A) = p_1 = 0.5$. If $A = \{H, T\}$, $P(A) = p_1 + p_2 = 1$.

**Steps for Calculating Probabilities of Events**

1. Define the experiment; that is, describe the process used to make an observation and the type of observation that will be recorded.
2. List the sample points.
3. Assign probabilities to the sample points.
4. Determine the collection of sample points contained in the event of interest.
5. Sum the sample point probabilities to get the probability of the event.

Counting the number of sample points in a sample space can help us to assign the probability. A method of determining the number of sample points for an experiment is to use combinatorial mathematics.

**Permutations Rule**
An ordered arrangement of $k$ objects taken from a set of $n$ objects ($1 \leq k \leq n$) is a permutation:

$$P_n^r = n(n-1) \cdots (n-r+1)$$

**Combinations Rule**
An unordered arrangement of $k$ objects taken from a set of $n$ objects is a combination:

$$C_n^r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where

$$n! = n(n-1)(n-2) \cdots (3)(2)(1)$$

and similarly for $r!$ and $(n-r)!$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Note: $0!$ is defined to be equal to 1.

**3.2 Unions and Intersections**

**Definition 3.5** The union of two events $A$ and $B$ is the event that occurs if either $A$ or $B$ (or both) occurs on a single performance of the experiment. We denote the union of events $A$ and $B$ by the symbol $A \cup B$. $A \cup B$ consists of all the sample points that belong to $A$ or $B$ or both.
Definition 3.6  The intersection of two events $A$ and $B$ is the event that occurs if both $A$ and $B$ occur on a single performance of the experiment. We write $A \cap B$ for the intersection of $A$ and $B$. $A \cap B$ consists of all the sample points belonging to both $A$ and $B$.

3.3 Complementary Events

Definition 3.7  The complement of an event $A$ is the event that $A$ does not occur - that is, the event consisting of all sample points that are not in event $A$. We denote the complement of $A$ by $A^c$.

Rule of Complements

The sum of the probabilities of complementary events equals 1; that is

\[ P(A) + P(A^c) = 1 \]

Example 7  Let $\#A$ denote the number of sample points in the event $A$ (with a similar definition for $\#B$, $\#(A \cup B)$, etc).

Given that $\#(A \cup B) = 20$, $\#(A) = 12$, $\#(B) = 9$ find:

(a) $\#(A \cap B)$

(b) $\#(A \cap B^c)$

(c) $\#(B \cap A^c)$

3.4 The Additive Rule and Mutually Exclusive Events

Additive Rule of Probability

The probability of the union of events $A$ and $B$ is the sum of the probability of event $A$ and the probability of event $B$, minus the probability of the intersection of event $A$ and $B$; that is

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

Definition 3.8  Events $A$ and $B$ are mutually exclusive if $A \cap B$ contains no sample points - that is if $A$ and $B$ have no sample points in common. For mutually exclusive events,

\[ P(A \cap B) = 0 \]
Probability of Union of Two Mutually Exclusive Events

If two events $A$ and $B$ mutually exclusive, the probability of the union of $A$ and $B$ equals the sum of the probability of $A$ and the probability of $B$; that is, $P(A \cup B) = P(A) + P(B)$.

3.5 Conditional Probability

If we have additional knowledge that might effect the outcome of an experiment, so we may need to alter the probability of an event of interest. A probability that reflects such additional knowledge is called the conditional probability of the event.

Conditional Probability Formula

To find the conditional probability that event $A$ occurs given that event $B$ occurs, divide the probability that both $A$ and $B$ occur by the probability that $B$ occurs; that is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where we assume that $P(B) \neq 0$

3.6 The Multiplicative Rule and Independent Events

Multiplicative Rule of Probability

$$P(A \cap B) = P(A)P(B|A)$$ or, equivalently, $P(A \cap B) = P(B)P(A|B)$

Definition 3.9 A partition is a collection of mutually exclusive sets (events) that together form the sample space.

![Diagram of a partition](image)

Law of total probability

The law of total probability states that given a partition $\{A_1, A_2, \ldots, A_k\}$ of the event space,

$$P(B) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$$

Definition 3.10 Events $A$ and $B$ are independent events if the occurrence of $B$ does not alter the probability that $A$ has occurred; that is, event $A$ and $B$ are independent if
\[ P(A|B) = P(A) \]

When events \( A \) and \( B \) are independent, it is also true that

\[ P(B|A) = P(B) \]

Events that are not independent are said to be **dependent**.

**Probability of Intersection of Two Independent Events**

If *events* \( A \) and \( B \) *are independent*, then the probability of the intersection of \( A \) and \( B \) equals the product of the probabilities of \( A \) and \( B \); that is

\[ P(A \cap B) = P(A)P(B) \]

The converse is also true: If \( P(A \cap B) = P(A)P(B) \), then events \( A \) and \( B \) are independent.

**Example 8** If \( P(A) = 0.4 \), \( P(B) = 0.5 \) and \( P(A \cup B) = 0.8 \) then what is \( P(A \cap B) \)? Are \( A \) and \( B \) independent?