Statistics 120 Spring Quarter 2008

Answers to HW # 24 (Chapter 11)

1.5 From $y = e^x$, we get $x = \log y$, $\frac{dx}{dy} = \frac{1}{y}$, and then $f_Y(y) = \frac{1}{(\beta-\alpha)y}$, $e^\alpha < y < e^\beta$. From $z = \log x$, we get $x = e^z$, $\frac{dx}{dz} = e^z$, and $f_Z(z) = \frac{e^z}{\beta-\alpha}$, $\log\alpha < z < \log\beta$. ■

1.7 From $y = \frac{1}{x}$ $(x \neq 0)$, we have $x = \frac{1}{y}$, $\frac{dx}{dy} = -\frac{1}{y^2}$, and therefore $f_Y(y) = \frac{1}{y^2} \times \frac{1}{\sqrt{2\pi}} y^2 e^{-\frac{y^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, y \in \Re$, which is the p.d.f. of the $N(0,\ 1)$ distribution. ■

2.6 (i) From $u = \frac{1}{\sqrt{2}}(x+y), v = \frac{1}{\sqrt{2}}(x-y)$, we get: $x + y = \sqrt{2}u, x - y = \sqrt{2}v$, and hence $x = \frac{\sqrt{2}}{2}(u+v), y = \frac{\sqrt{2}}{2}(u-v)$. Furthermore,

$$
\begin{aligned}
J &= \begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = -1, \quad \text{so that} \\
f_{U,V}(u,v) &= \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left\{\left[\frac{\sqrt{2}}{2}(u+v)\right]^2 + \left[\frac{\sqrt{2}}{2}(u-v)\right]^2\right\}\right] \\
&= \frac{1}{2\pi} \exp\left[-\frac{1}{2}(u^2+v^2)\right] \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}, \quad u, v \in \Re.
\end{aligned}
$$

(ii) From the last expression above, it follows that $U \sim N(0,\ 1)$ and $V \sim N(0,\ 1)$.

(iii) From the factorization in the same expression above, it also follows that U and V are independent.

(iv) By independence of X and Y, we have directly:

$$X + Y \sim N(0,\ 2) \text{ and hence } U = \frac{1}{\sqrt{2}}(X+Y) \sim N(0,\ 1). \text{ Likewise,}$$

$$X - Y \sim N(0,\ 2) \text{ and hence } V = \frac{1}{\sqrt{2}}(X-Y) \sim N(0,\ 1). \quad ■$$

5.1 (i) For $x \geq 1, F(x) = \int_1^x c t^{-(c+1)} dt = -\int_1^x dt^{-c} = -t^{-c}|_1^x = 1 - \frac{1}{x^c}$. So,

$$F(x) = 0 \text{ for } x \leq 1, \quad \text{and} \quad F(x) = 1 - \frac{1}{x^c} \text{ for } x > 1.$$

(ii) In formulas (11.30) and (11.31), take $a = 1, b = \infty$ to obtain.

$$f_U(u) = n \times \left(\frac{1}{u^c}\right)^{n-1} \times cu^{-(c+1)} = n\,cu^{-n\,c-1}, \quad u > 1,$$

and

$$f_V(v) = n\left(1 - \frac{1}{v^c}\right)^{n-1} \times cv^{-(c+1)} = n\,c(v^c - 1)^{n-1} v^{-n\,c-1}, \quad v > 1. \quad ■$$

5.7 Clearly, P(system works beyond t) $= P[\min(X,Y) > t]$. So, setting $T = \min(X,Y)$, Example 14 gives $f_T(t) = (2\lambda) \times e^{-(2\lambda)t}, t > 0$ Hence:

(i) $P(T > t) = \int_t^\infty (2\lambda)e^{-(2\lambda)x}dx = -e^{-(2\lambda)x}|_t^\infty = e^{-2\lambda t}$.

(ii) Since the distribution is the Negative Exponential with parameter 2λ, we have $ET = 1/2\lambda$.

(iii) For $\lambda = \frac{1}{3}$, $P(T > t) = e^{-2t/3}$ (and for $t = 3$, e.g., this probability is $e^{-2} \simeq 0.135$), and $ET = 1.5$. ■