Statistics 120 Spring Quarter 2008

Answers to HW # 23 (Chapter 10)

1.1 For $x = 0$ and $y = 1$, we have: $f_{X,Y}(0,1) = \frac{1}{8}, f_X(0) = \frac{3}{16}$, and $f_Y(1) = \frac{1}{2}$. Since $f_{X,Y}(0,1) = \frac{1}{8} \neq \frac{3}{16} \times \frac{1}{2} = f_X(0)f_Y(1)$, the r.v.'s X and Y are not independent. ■

1.3 The points at which $f_{X,Y,Z}$ is not zero (and, in fact, equals 1/4) are the points:

$$(1,0,0), \quad (0,1,0), \quad (0,0,1), \quad (1,1,1).$$

Then:

(i)

$$f_{X,Y}(1,0) = f_{X,Y}(0,1) = f_{X,Y}(0,0) = f_{X,Y}(1,1) = 1/4;$$
$$f_{X,Z}(1,0) = f_{X,Z}(0,1) = f_{X,Z}(0,0) = f_{X,Y}(1,1) = 1/4;$$
$$f_{Y,Z}(1,0) = f_{Y,Z}(0,1) = f_{Y,Z}(0,0) = f_{Y,Z}(1,1) = 1/4.$$

(ii) From part (i):

$$f_X(1) = f_X(0) = \frac{1}{2}; \quad f_Y(1) = f_Y(0) = \frac{1}{2}; \quad f_Z(1) = f_Z(0) = \frac{1}{2}.$$

(iii) From parts (i) and (ii), we have that, for all x, y, and z, $f_X(x)f_Y(y) = f_{X,Y}(x,y)$; $f_X(x)f_Z(z) = f_{X,Z}(x,z)$; and $f_Y(y)f_Z(z) = f_{Y,Z}(y,z)$, so that the pairs X, Y; X, Z; and Y, Z consist of independent r.v.'s.

(iv) The r.v.'s X, Y, and Z are not independent, because, for example,

$$f_X(1)f_Y(1)f_Z(1) = \frac{1}{8} \neq \frac{1}{4} = f_{X,Y,Z}(1,1,1). \quad ■$$

1.9 (i) We have: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy = c\int_{(x^2+y^2\leq 9)}\int dx\,dy = c.9\pi$, because the integral is the area of a circle centered at the origin and having radius $r = 3$, so that the area is $\pi r^2 = 9\pi$. Hence $c = 1/9\pi$.

(ii) $f_X(x) = c\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy = \frac{2}{9\pi}\sqrt{9-x^2}, 0 \leq x \leq 3$, and likewise $f_Y(y) = \frac{2}{9\pi}\sqrt{9-y^2}, 0 \leq y \leq 3$.

(iii) For example, for $x=0, y=3$, $f_X(0) = 2/3\pi$; $f_Y(3) = 0$ and $f_{X,Y}(0,3) = \frac{1}{9\pi}$, so that $f_X(0)f_Y(3) \neq f_{X,Y}(0,3)$, and the r.v.'s X and Y are dependent. ■

1.13 (i) For $y > 0$,

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(-\log X \leq y) = P(\log X \geq -y) \\ &= P(X \geq e^{-y}) = 1 - P(X < e^{-y}) \\ &= 1 - P(X \leq e^{-y}) = 1 - e^{-y}; \quad \text{i.e.,} \end{aligned}$$

$F_Y(y) = 1 - e^{-y}$, for $y > 0$, and $F_Y(y) = 0$, for $y \leq 0$. Hence, for $y > 0, f_Y(y) = \frac{d}{dy}(1 - e^{-y}) = e^{-y}$, so that Y has the

Negative Exponential distribution with parameter $\lambda = 1$. Or, to put it differently, it has the Gamma distribution with $\alpha = 1$ and $\beta = 1/\lambda$.

(ii) At this point, recall that, if the r.v. U has the Gamma distribution with parameters α and β, then its m.g.f. is:

$$M_U(t) = \frac{1}{(1-\beta t)^\alpha}, \quad t < \frac{1}{\beta}.$$

In the case of the Negative Exponential distribution with parameter $\lambda = 1$ (i.e., $\alpha = 1$ and $\beta = \frac{1}{\lambda} = 1$), the m.g.f. becomes:

$$M_U(t) = \frac{1}{1-t}, \quad t < 1.$$

Therefore, by independence of the r.v.'s $Y_1, \ldots, Y_n$, we have:

$$M_Z(t) = \frac{1}{(1-t)^n}, \quad t < 1,$$

which is the m.g.f. of Gamma with $\alpha = n$ and $\beta = 1$. ■

1.23 Since $E\bar{X} = \mu$ and $Var(\bar{X}) = \sigma^2/n$, the Tchebichev inequality gives:

$$P(|\bar{X} - \mu| < 0.5\sigma) \geq 1 - \frac{\sigma^2/n}{(0.5\sigma)^2} = 1 - \frac{1}{0.25n} = 1 - \frac{4}{n}.$$

Thus, it suffices

$$1 - \frac{4}{n} \geq 0.99, \quad \text{or} \quad n \geq 400, \text{ so that } n = 400. \quad ■$$

1.29 (i) By independence of X and Y, $f_{X,Y}(x,y) = \lambda_1\lambda_2 e^{-\lambda_1 X - \lambda_2 Y}$, $x > 0, y > 0$, so that:

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_0^y \lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2 y} dx dy \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 y} (\int_0^y \lambda_1 e^{-\lambda_1 x} dx) dy \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 y} (-e^{-\lambda_1 x}|_0^y) dy \\ &= \int_0^\infty \lambda_2 e^{-\lambda_2 y} (1 - e^{-\lambda_1 y}) dy \\ &= 1 - \tfrac{\lambda_2}{\lambda_1+\lambda_2} \int_0^\infty (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2) y} dy \\ &= 1 - \tfrac{\lambda_2}{\lambda_1+\lambda_2}; \text{ i.e., } P(X < Y) = \tfrac{\lambda_1}{\lambda_1+\lambda_2}. \end{aligned}$$

(ii) Here $P(X < Y) = \frac{\lambda}{4\lambda} = \frac{1}{4} = 0.25$. ■