Statistics 120 Spring Quarter 2008

Answers to HW #14 (Chapter 6)

2.4 We have:

(i) (a) $g(X) = cX$, so that $Eg(X) = cEX = c/\lambda$.
(b) $g(X) = c(1 - 0.5e^{-\alpha X})$, so that:

$$\begin{aligned} Eg(X) &= \int_0^\infty c(1-0.5e^{-\alpha x})\lambda e^{-\lambda x}dx = c\int_0^\infty \lambda e^{-\lambda x}dx \\ &\quad -0.5c\lambda \int_0^\infty e^{-(\alpha+\lambda)x}dx \\ &= c - \frac{0.5c\lambda}{\alpha+\lambda}\int_0^\infty (\alpha+\lambda)e^{-(\alpha+\lambda)x}dx \\ &= c - \frac{0.5c\lambda}{\alpha+\lambda} = \frac{(\alpha+0.5\lambda)c}{\alpha+\lambda}. \end{aligned}$$

(ii) (a) 10; (b) 1.5. ■

2.5 Here $f(x) = \lambda e^{-\lambda x}, x > 0$, so that:

(i) $F(x) = \int_0^x \lambda e^{-\lambda t}dt = -e^{-\lambda t}|_0^x = 1 - e^{-\lambda x}$, and $1 - F(x) = e^{-\lambda x}$. Therefore

$$r(x) = \frac{f(x)}{1-F(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda, \text{ constant for all } x > 0.$$

(ii)
$$\begin{aligned} P(X > s+t | X > s) &= \frac{P(X > s+t, X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)} \\ &= \frac{1-F(s+t)}{1-F(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} = P(X>t), \text{ independent of } s > 0. \end{aligned}$$

The independence from s says that the underlying distribution is "memoryless." This is not a particularly desirable attribute if this distribution is to serve as a lifetime distribution. ■

2.6 Indeed, $P(T > t) = P(0 \text{ events occurred in the time interval } (0,t]) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$. So, $1 - F_T(t) = e^{-\lambda t}, t > 0$, and hence $f_T(t) = \lambda e^{-\lambda t}, t > 0$, and T is as described. ■

2.7 By Exercise 2.6, the distribution of the waiting time is a r.v. Y which is distributed as Negative Exponential with parameter λt. Here $\lambda = 3$ and $t = 1$ second. Then:

(i) $P(\text{1st particle will arrive within 1 second}) = P(Y \le 1) = \int_0^1 3e^{-3x}dx = -e^{-3x}|_0^1 = 1 - e^{-3} \simeq 0.95$.

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(ii) P(waiting for at least another second, given that we have already waited 1 second) $= P(T \geq 2 | T > 1) = \frac{P(T \geq 2, T > 1)}{P(T > 1)} = \frac{P(T \geq 2)}{P(T > 1)} = \frac{e^{-6}}{e^{-3}} = \frac{1}{e^3} \simeq 0.05$. ■

2.8 (i) Observe that:

$$\int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} dx = \int_0^\infty e^{-\alpha x^\beta}(\alpha\beta x^{\beta-1})dx = -\int_0^\infty de^{-\alpha x^\beta}$$
$$= -e^{-\alpha x^\beta}\big|_0^\infty = -\frac{1}{e^{\alpha x^\beta}}\bigg|_0^\infty = -0 + 1 = 1.$$

(ii) For $\beta = 1$ and any $\alpha > 0$.

(iii) For $n = 1, 2, \ldots$,

$$EX^n = \alpha\beta \int_0^\infty x^n \times x^{\beta-1} e^{-\alpha x^\beta} dx = \alpha\beta \int_0^\infty x^{n+\beta-1} e^{-\alpha x^\beta} dx.$$

Set $\alpha x^\beta = t$, so that $x = \frac{t^{1/\beta}}{\alpha^{1/\beta}}$, $dx = \frac{t^{\frac{1}{\beta}-1}}{\beta\alpha^{1/\beta}} dx$ and $0 < t < \infty$. Then:

$$EX^n = \alpha\beta \int_0^\infty \frac{t^{\frac{n+\beta-1}{\beta}}}{\alpha^{\frac{n+\beta-1}{\beta}}} \times \frac{t^{\frac{1}{\beta}-1}}{\beta\alpha^{1/\beta}} e^{-t} dt$$
$$= \frac{\Gamma(\frac{n}{\beta}+1)}{\alpha^{n/\beta}} \int_0^\infty \frac{1}{\Gamma(\frac{n}{\beta}+1)} t^{(\frac{n}{\beta}+1)-1} e^{-t} dt = \frac{\Gamma(\frac{n}{\beta}+1)}{\alpha^{n/\beta}}$$

because $\frac{1}{\Gamma(\frac{n}{\beta}+1)} t^{(\frac{n}{\beta}+1)-1} e^{-t} (t > 0)$, is the p.d.f. of the Gamma distribution with parameters $\frac{n}{\beta}+1$ and 1. Thus, $EX^n = \Gamma(\frac{n}{\beta}+1)/\alpha^{n/\beta}$. For $n = 1$ and $n = 2$, we get $EX = \Gamma(\frac{1}{\beta}+1)/\alpha^{1/\beta}$ and $EX^2 = \Gamma(\frac{2}{\beta}+1)/\alpha^{2/\beta}$, respectively, so that

$$Var(X) = \frac{\Gamma(\frac{2}{\beta}+1)}{\alpha^{2/\beta}} - \left[\frac{\Gamma(\frac{1}{\beta}+1)}{\alpha^{1/\beta}}\right]^2 = \frac{\Gamma(\frac{2}{\beta}+1) - \left[\Gamma(\frac{1}{\beta}+1)\right]^2}{\alpha^{2/\beta}}. \quad ■$$

2.9 (i) From Exercise 2.8(i),

$$F(x) = \int_0^x \alpha\beta t^{\beta-1} e^{-\alpha t^\beta} dt = -e^{-\alpha t^\beta}\big|_0^x = 1 - e^{-\alpha x^\beta},$$

so that, for $x > 0$,

$$1 - F(x) = e^{-\alpha x^\beta} \text{ and } r(x) = \frac{f(x)}{1 - F(x)} = \frac{\alpha\beta x^{\beta-1} e^{-\alpha x^\beta}}{e^{-\alpha x^\beta}} = \alpha\beta x^{\beta-1}.$$

(ii)
$$P(X > s + t | X > s) = \frac{P(X > s + t, X > s)}{P(X > s)} = \frac{P(X > s + t)}{P(X > s)}$$
$$= \frac{1 - F(s + t)}{1 - F(s)} = \frac{e^{-\alpha(s+t)^\beta}}{e^{-\alpha s^\beta}} = e^{-\alpha[(s+t)^\beta - s^\beta]}.$$

(iii) Here, both the failure rate and the conditional survival probability do depend on the variables involved. This *is* a desirable characteristic for lifetime distributions. ■