Statistics 120 Spring Quarter 2008

Answers to HW # 11 (Chapter 5)

2.1 (i) $EX = -1 \times \frac{1}{18} + 0 \times \frac{8}{9} + 1 \times \frac{1}{18} = 0, EX^2 = \frac{1}{9}$, so that $Var(X) = \frac{1}{9}$ and s.d. of $X = \frac{1}{3}$, So, $\mu = 0, \sigma = 1/3$.

(ii) $P(|X - \mu| \geq k\sigma) = P(|X| \geq \frac{k}{3}) = \frac{2}{18} = \frac{1}{9} \simeq 0.111$, both for $k = 2$ and $k = 3$.

(iii) $P(|X - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$. For $k = 2, \frac{1}{k^2} = 0.25$ (more than twice as large the exact probability), and for $k = 3, \frac{1}{k^3} = \frac{1}{9}$ (exactly the same as the exact probability). ■

2.3 From Exercise 3.11(i) in Chapter 3, $c = 3/4$. Next,

(i)
$$EX = c\int_{-1}^{1} x(1-x^2)dx = c\left(\frac{x^2}{2}\Big|_{-1}^{1} - \frac{x^4}{4}\Big|_{-1}^{1}\right) = 0,$$

$$EX^2 = c\int_{-1}^{1} x^2(1-x^2)dx = c\left(\frac{x^3}{3}\Big|_{-1}^{1} - \frac{x^5}{5}\Big|_{-1}^{1}\right) = \frac{4c}{15} = \frac{4}{15} \times \frac{3}{4} =$$

$$\frac{1}{5} = 0.2, \text{ so that } Var(X) = 0.2.$$

(ii) $P(-0.9 < X < 0.9) = P(|X| < 0.9) = P(|X - EX| < 0.9) \geq 1 - \frac{0.2}{(0.9)^2} \simeq 1 - 0.247 = 0.753$.

In Exercise 3.11(ii) of Chapter 3, it was seen that the exact probability $P(-0.9 < X < 0.9) = 0.9855$. Thus, the approximate probability 0.753 is about 76.4% of the exact probability. ■

3.1 (i) $EX = \int_0^1 x \times 3x^2 dx = \frac{3}{4} = 0.75, m = \int_0^{x_{0.5}} 3x^2 dx = x_{0.5}^3 = 0.5$, so that $x_{0.5} = (0.5)^{1/3} \simeq 0.794$, and mean < median.

(ii) $\int_0^{x_{0.125}} 3x^2 dx = x_{0.125}^3 = 0.125 = \frac{1}{8} = \frac{1}{2^3}$, so that $x_{0.125} = \frac{1}{2} = 0.5$. ■

3.2 (i) From $P(X \leq x_p) = \int_0^{x_p} x^n dx = \frac{1}{n+1} x^{n+1}\Big|_0^{x_p} = \frac{x_p^{n+1}}{n+1} = p$, we get $x_p = [(n+1)p]^{1/(n+1)}$.

(ii) For $p = 0.5$ and $n = 3$, we have $x_{0.5} = 2^{1/4} \simeq 1.189$. ■

3.3 (i) From $P(X \leq x_p) = \int_0^{x_p} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^{x_p} = 1 - e^{-\lambda x_p} = p$, we get $e^{-\lambda x_p} = 1 - p$, or $x_p = -\frac{\log(1-p)}{\lambda}$.

(ii) For $\lambda = 1/10$ and $p = 0.25$, we have: $x_{0.25} = -10\log(0.75) \simeq 2.877$. ■

3.5 (i) By differentiation, we have $f(x) = \frac{3}{4}(2x - x^2), 0 < x \leq 2$.
Then, for $0 < x < 2$:

(ii) $f'(x) = 0$ yields $x = 1$ and $f''(1) = -\frac{3}{2} < 0$, so that $m = 1$ is the (unique) mode of f.

(iii) For simplicity, set $c = x_{5/32}$. Then, from $\int_0^c \frac{3}{4}(2x - x^2)dx = \frac{5}{32}$, we obtain, after some simplifications, $8c^3 - 24c^2 + 5 = 0$. Observe that $c = \frac{1}{2}$ is a root of this equation. Thus, $x_{5/32} = \frac{1}{2}$. (The other roots are $\simeq 2.927$ and $\simeq -0.427$, outside $(0, 2]$). ■

3.7 (i) Mode = 1 and the maximum is 1/2.

(ii) Mode = 1 and the maximum is $1 - \alpha$; $\alpha = \frac{1}{2}$.

(iii) Mode = 0 and the maximum is 2/3. ■