Statistics 120 Spring Quarter 2008

Answers to HW # 10 (Chapter 5)

1.1 In Exercise 3.1 in Chapter 3:

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\begin{aligned} EX &= 1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=\frac{12}{8}=\frac{3}{2}=1.5,\\ EX^2 &= 1\times\frac{3}{8}+4\times\frac{3}{8}+9\times\frac{1}{8}=\frac{24}{8}=3, \text{ so that}\\ Var(X) &= 3-1.5^2=0.75 \quad \text{and s.d. of } X\simeq 0.866. \quad \blacksquare \end{aligned}$$

1.3 (i) Here $EX = 1\times 0.2 + 2\times 0.4 + 3\times 0.3 + 4\times 0.1 = 2.3, EX^2 = 1\times 0.2+4\times 0.4+9\times 0.3+16\times 0.1 = 6.1$, so that $Var(X) = 0.81$.

(ii) Let Y be the r.v. denoting the quantity described. Then $Y = 100 - 5X$ and therefore $EY = 100 - 5\times 2.3 = 88.5$, $Var(Y) = Var(5X) = 25\times 0.81 = 20.25$, and s.d. of $Y = 4.5$. $\blacksquare$

1.5 The values of X are: 1, 2, 3, 4, 5, 6, each taken with probability 1/6. Thus:

(i) $$M_X(t)=\sum_{x=1}^{6} e^{tx}\times\frac{1}{6}=\frac{1}{6}(e^t+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t}),\quad t\in\Re.$$

(ii) $$\begin{aligned} EX &= \frac{d}{dt}M_X(t)\Big|_{t=0}=\frac{1}{6}(e^t+2e^{2t}+3e^{3t}+4e^{4t}+5e^{5t}+6e^{6t})\Big|_{t=0}\\ &= \frac{1}{6}(1+2+3+4+5+6)=\frac{21}{6}=3.5;\\ EX^2 &= \frac{d^2}{dt^2}M_X(t)\Big|_{t=0}=\frac{1}{6}(e^t+4e^{2t}+9e^{3t}+16e^{4t}+25e^{5t}+36e^{6t})\Big|_{t=0}\\ &= \frac{1}{6}(1+4+9+16+25+36)=\frac{91}{6}, \quad \text{so that}\\ Var(X) &= \frac{91}{6}-\left(\frac{21}{6}\right)^2=\frac{105}{36}, \quad \text{and} \quad \text{s.d. of } X=\frac{\sqrt{105}}{6}\simeq 1.708. \quad \blacksquare \end{aligned}$$

1.7 We have:

(i) $E[X(X-1)] = E(X^2 - X) = EX^2 - EX$, so that:

$$EX^2 = E[X(X-1)] + EX = 27.5 + 5 = 32.5.$$

(ii) $Var(X) = EX^2 - (EX)^2 = 32.5 - 25 = 7.5$, and s.d. of $X = \sqrt{7.5} \simeq 2.739$. $\blacksquare$

1.9 We have:

(i) By #5 in Table 6 in the Appendix, applied with $r = 1/3$, we have:

$$EX = \sum_{x=1}^{\infty} x \times \frac{2}{3}\left(\frac{1}{3}\right)^x = \frac{2}{3}\sum_{x=1}^{\infty} x\left(\frac{1}{3}\right)^x = \frac{2}{3} \times \frac{1/3}{(1-1/3)^2} = \frac{1}{2}.$$

(ii) By #4 in Table 6 in the Appendix, applied with $r = e^t/3$ ($t < \log 3$), we have:

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \times \frac{2}{3}\left(\frac{1}{3}\right)^x = \frac{2}{3}\sum_{x=0}^{\infty}\left(\frac{e^t}{3}\right)^x = \frac{2}{3} \times \frac{1}{1-\frac{e^t}{3}} = \frac{2}{3-e^t},$$

provided $\frac{e^t}{3} < 1$ or $t < \log 3$.

(iii) $EX = \frac{d}{dt}\left(\frac{2}{3-e^t}\right)\Big|_{t=0} = \frac{2e^t}{(3-e^t)^2}\Big|_{t=0} = \frac{1}{2}$. ■

1.11 Here $EX = \int_0^1 x(3x^2 - 2x + 1)dx = \frac{3}{4}x^4|_0^1 - \frac{2}{3}x^3|_0^1 + \frac{x^2}{2}|_0^1 = \frac{7}{12}$, $EX^2 = \int_0^1 x^2(3x^2 - 2x + 1)dx = \frac{3}{5}x^5|_0^1 - \frac{1}{2}x^4|_0^1 + \frac{1}{3}x^3|_0^1 = \frac{13}{30}$, so that $Var(X) = \frac{13}{30} - \frac{49}{144} = \frac{67}{720} \simeq 0.093$. ■

1.13 We have:

$$\begin{aligned} EX &= \int_0^{\infty} x \times \lambda^2 x e^{-\lambda x} dx = -\lambda x^2 e^{-\lambda x}|_0^{\infty} + 2\lambda\int_0^{\infty} x e^{-\lambda x} dx \\ &= 2\lambda\int_0^{\infty} x e^{-\lambda x} dx = 2\int_0^{\infty} x \times \lambda e^{-\lambda x} dx = \frac{2}{\lambda}. \end{aligned}$$ ■

1.17 Indeed,

$$M_X(t) = Ee^{tX} = E\left(\sum_{n=0}^{\infty}\frac{(tX)^n}{n!}\right) = \sum_{n=0}^{\infty} E\frac{(tX)^n}{n!}$$

(assuming that the interchange of the expectation and the summation sign is allowed)

$$= \sum_{n=0}^{\infty}(EX^n)\frac{t^n}{n!}.$$ ■

1.21 (i) From $\int_0^{\infty}\frac{c\,dx}{(1+x)^4} = c\int_0^{\infty}(1+x)^{-4}d(1+x) = -\frac{c}{3}x(1+x)^{-3}\,|_0^{\infty} = \frac{c}{3} = 1$, we get $c = 3$.

(ii) $P(1 \le X \le 4) = 3\int_1^4 \frac{dx}{(1+x)^4} = -\frac{1}{(1+x)^3}\,|_1^4 = \frac{1}{8} - \frac{1}{125} = \frac{117}{1{,}000} = 0.117$.

(iii) $EX = \int_0^{\infty}\frac{3x dx}{(1+x)^4} = 3\int_0^{\infty}(1+x)^{-4}x dx = -\int_0^{\infty} x d(1+x)^{-3}$
$= -\frac{x}{(1+x)^3}\,|_0^{\infty} + \int_0^{\infty}\frac{dx}{(1+x)^3} = \int_0^{\infty}(1+x)^{-3}dx$
$= -\frac{1}{2}\int_0^{\infty} d(1+x)^{-2} = -\frac{1}{2}\frac{1}{(1+x)^2}\,|_0^{\infty} = \frac{1}{2}$. ■