Instructions: Solve the problems in the blank space below each, continuing on the back of the sheet as needed. Closed Book. You may consult two, double-sided, letter-size sheets of personal notes and may use, without proof, results from the lectures and Lab #1.

1. Let $A$ be an $m \times n$ matrix of rank $r \geq 1$.
   
   a) (5 points) The null space $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$. Show that
   
   \[ \mathcal{N}(A) = \{(A^+ A - I_n)c : c \in \mathbb{R}^n\}. \]
   
   b) (5 points) The range space of any $n \times m$ matrix $B$ is $\mathcal{R}(B) = \{Bz : z \in \mathbb{R}^m\}$. Show that
   
   \[ \mathcal{R}(A^+) = \mathcal{R}(A'). \]
   
   c) (5 points) Show that
   
   \[ \mathcal{R}(A^+ A) = \mathcal{R}(A') \]
   
   and that $A^+ A$ is a symmetric, idempotent $n \times n$ matrix. What geometrical operation in $\mathbb{R}^n$ does $A^+ A$ accomplish?

1. a) Equation $Ax = 0$ is consistent with solution $x = 0$.
   
   \[ \mathcal{N}(A) = \{\text{all solutions to } Ax = 0\} = A^+ : 0 + \{(A'^- A - I_n)c : c \in \mathbb{R}^n\} = \{(A^+ A - I_n)c : c \in \mathbb{R}^n\}. \]

   b) Let $A = U L V'$, with $U^T U = V'^T V = I_n$, $L = \text{diag}(\lambda_i)$, $\lambda_i \geq 0$, $i \geq 1$, $\sum \lambda_i > 0$.
   
   \[ A'^T = V L^{-1} U' \Rightarrow \mathcal{R}(A') \subset \mathcal{R}(V) \Rightarrow \mathcal{R}(A') = \mathcal{R}(V) \]
   
   \[ A'^+ U L = V \Rightarrow \mathcal{R}(V) \subset \mathcal{R}(A') \]

   Similarly, $A' = V L U' \Rightarrow \mathcal{R}(A') \subset \mathcal{R}(V) \Rightarrow \mathcal{R}(A') = \mathcal{R}(V)$.

   Hence $\mathcal{R}(A^+ A) = \mathcal{R}(V) = \mathcal{R}(A')$.

   c) $\mathcal{R}(A^+ A) \subset \mathcal{R}(A')$ and $\mathcal{R}(A^+) = \mathcal{R}(A^+ A A^+) < \mathcal{R}(A^+ A)$.

   Hence $\mathcal{R}(A^+ A) = \mathcal{R}(A^+) = \mathcal{R}(A')$ using part b.

   $A^+ A$ is symmetric by characterization of $A^+$.

   $A^+ A$ is idempotent because $A^+ A A^+ A = A^+ A$.

   $A^+ A$ is the orthogonal projection into $\mathcal{R}(A')$. 


2. Consider the Gaussian linear model in which \( y = (y_1, y_2, y_3, y_4)' \) has a \( N(\eta, \sigma^2 I_4) \) distribution and \( \eta = (\eta_1, \eta_2, \eta_3, \eta_4)' \) has the structure

\[
\begin{align*}
\eta_1 &= m + b, & \eta_2 &= m - b + c, \\
\eta_3 &= m + b, & \eta_4 &= m - b + c.
\end{align*}
\]

In this model, \( m, b, c \) and \( \sigma^2 > 0 \) are unknown real-valued parameters.

a) (5 points) Let \( \beta = (m, b, c)' \). Show that \( \hat{\beta} = (1/4)(y_1 + y_2 + y_3 + y_4, y_1 - y_2 + y_3 - y_4, 0)' \) is a least squares estimator of \( \beta \).

b) (5 points) Show that \( m \) does not have a unique least squares estimator.

\[ a) \ \eta = X\beta \text{ with } X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \text{ so } X'X = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & -2 \\ 2 & -2 & 2 \end{pmatrix} \]

Normal equation \( X'X\beta = X'y \) is here

\[
\begin{align*}
4m + 2c &= y_1 + y_2 + y_3 + y_4 \\
4b - 2c &= y_1 - y_2 + y_3 - y_4 \\
2m - 2b + 2c &= y_2 - y_4
\end{align*}
\]

The stated \( \hat{\beta} = (\hat{m}, \hat{b}, \hat{c})' \) solves this.

b) \( \lambda'\beta \) has unique LSE \( \iff \lambda' = \alpha'X \) for some \( \alpha \in \mathbb{R}^4 \).

Here \( \lambda'\beta = m \iff \lambda' = (1, 0, 0) \)

\[
X'\alpha = (1, 0, 0) \iff \begin{cases} a_1 + a_2 + a_3 + a_4 = 1 \\ a_1 - a_2 + a_3 - a_4 = 0 \\ a_2 + a_4 = 0 \end{cases}
\]

The 2nd and 3rd equations yield \( a_1 + a_2 = a_3 + a_4 = 0 \) which contradicts the 1st equation.

Hence \( X'\alpha = 0 \) is not possible \( \iff m \) does not have unique LSE.
3. Consider the Gaussian linear model

\[ y_i = \beta_1 + \beta_2 x_i + e_i, \quad 1 \leq i \leq n, \]

where the \( \{x_i\} \) are distinct known values such that \( \sum_{i=1}^n x_i = 0 \) and the \( \{e_i\} \) are independent, identically distributed \( N(0, \sigma^2) \) random variables. Here \( \beta_1, \beta_2 \) and \( \sigma^2 > 0 \) are unknown real-valued parameters. Your answers to the following questions are to be simplified algebraically to elementary forms that are convenient for use on a basic scientific calculator.

a) (5 points) Let \( \beta = (\beta_1, \beta_2)' \). Find from general theory and simplify the least squares estimator \( \hat{\beta} \) of \( \beta \). Give the distribution of \( \hat{\beta} \) in simplest form.

b) (5 points) Find from general theory and simplify the least squares estimator \( \hat{\sigma}^2 \) of \( \sigma^2 \). Give the distribution of \( \hat{\sigma}^2 \) in simplest form.

c) (5 points) Obtain from general least squares theory and then simplify the F-statistic for testing \( H: \beta_1 = 0, \beta_2 \in R, \sigma^2 > 0 \) versus the alternatives \( K: \beta_1 \neq 0, \beta \in R^2, \sigma^2 > 0 \). Give in simplest form the distribution of this F-statistic under the null hypothesis.

a) \( y = X \beta + e \) with \( X' = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \end{pmatrix} \) and \( X'X = \begin{pmatrix} n & 0 \\ 0 & \sum x_i^2 \end{pmatrix} \)

\( r = \text{rank}(X) = 2 \) because the \( \{x_i\} \) are distinct

LSE \( \hat{\beta} = (X'X)^{-1}X'y = \left( \bar{x}, \frac{\sum x_i y_i}{\sum x_i^2} \right) = \left( \hat{\beta}_1, \hat{\beta}_2 \right)' \), \( \bar{y} = \frac{1}{n} \sum y_i \)

\( \hat{\beta} \sim N(\beta, \sigma^2(X'X)^{-1}) = N\left( \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1/n & 0 \\ 0 & \sum x_i^2 \end{pmatrix} \right) \)

\( \hat{\beta}_1 \sim N(\beta_1, \sigma^2/n), \hat{\beta}_2 \sim N(\beta_2, \sigma^2/\sum x_i^2) \) \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) independent

b) \( \hat{\sigma}^2 = \frac{1}{n-r} \sum (y_i - \hat{\beta}_1 x_i - \hat{\beta}_2 x_i)^2 \) with \( x_i \) from part a

\( \frac{(n-2)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-2} \)

c) Part a gives the general model and \( \hat{\mu} = X \hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, x_1, \ldots, \hat{\beta}_2, x_n)' \)

Submodel under \( H: y = X_0 \hat{\beta} + e \), \( X_0 = (x_1, x_2, \ldots, x_n)' \)

\( r_0 = \text{rank}(X_0) = 1 \)

LSE under \( H \) is \( \hat{\beta}_0 = (X_0'X_0)^{-1}X_0'y = \frac{\sum x_i y_i}{\sum x_i^2} = \hat{\beta}_0 \) in part a

Thus \( \hat{\mu}_0 = X_0 \hat{\beta}_0 = (\hat{\beta}_1, \hat{\beta}_2, x_1, \ldots, \hat{\beta}_2, x_n)' \)

\( \hat{\mu} - \hat{\mu}_0 = (\bar{y}, \bar{y}, \ldots, \bar{y})' \), \( r_1 = r - r_0 = 2 - 1 = 1 \)

F statistic \( T = \frac{(\hat{\mu} - \hat{\mu}_0)'R^{-1}(\hat{\mu} - \hat{\mu}_0)}{\hat{\sigma}^2} = \frac{n \bar{y}^2}{\hat{\sigma}^2} \sim F_{1, n-2} \) under \( H \)