1. Let $A$ be an arbitrary $m \times n$ matrix of rank $r$. Show that its pseudoinverse $A^+$ has the following properties. State clearly the standard results from linear algebra used in your proofs.

a) If $A$ is square and has an inverse, then $A^+ = A^{-1}$.

b) If $c$ is any non-zero scalar, then $(cA)^+ = c^{-1}A^+$.

c) $(A^+)^+ = A$.

d) $(A^+) = (A^+)^+$.

e) $(A'A)^+ = A^+(A')^+ (AA')^+ = (A')^+A^+.$

f) $A^+A$ and $AA^+$ are each symmetric and idempotent matrices.

g) $A^+ = (A'A)^+ A' = A'(AA')^+$.

h) $A^+ = \lim_{\epsilon \to 0^+} (A'A + \epsilon I_m)^{-1}A' = \lim_{\epsilon \to 0^+} A'(AA' + \epsilon I_m)^{-1}.$

i) $\text{rank}(A) = \text{rank}(AA^+) = \text{tr}(AA^+) = \text{tr}(A^+A) = \text{rank}(A^+A) = \text{rank}(A^+).$

j) If $A$ is symmetric and positive semidefinite, then we may take $U = V$ in the singular value decomposition $A = ULV'$.

k) If $A$ is symmetric and idempotent, then $A^+ = A$.

l) Suppose $S$ is any $s \times m$ matrix, with $s \geq m$, such that $S'S = I_m$. Suppose $T$ is any $t \times n$ matrix, with $t \geq n$ such that $T'T = I_n$. Then $(SAT')^+ = TA^+S'$.

m) Suppose that $P$ is any $m \times m$ symmetric, idempotent matrix. Suppose that $Q$ is any $n \times n$ symmetric, idempotent matrix. Then $(PAQ)^+ = Q(PAQ)^+P$.

n) Suppose that $A$ is any $m \times n$ matrix. Let $\mathcal{R}(A) = \{Ax : x \in \mathbb{R}^n\}$ be the range of $A$. Then $\mathcal{R}(A) = \mathcal{R}(AA^+)$ and $\mathcal{R}(A^+) = \mathcal{R}(A^+A) = \mathcal{R}(A')$.

o) Suppose that the $m \times n$ matrix $A$ has the singular value decomposition $A = ULV'$, where $U'U = V'V = I_r$ and $r = \text{rank}(A)$. Then $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{R}(A') = \mathcal{R}(V)$.

2. Let $A$ be an arbitrary $m \times n$ matrix, $m \geq n$, that has mutually orthogonal, nonzero columns.

a) Find a singular value decomposition $A = ULV'$, expressing $U$, $L$, $V$ explicitly as simple algebraic functions of $A$.

b) Find algebraically the pseudoinverse $A^+$ as a simple function of $A$. Verify directly that your $A^+$ has the four properties that characterize a pseudoinverse.