5.20 (a) The possible values of $X$ are 0, 1, 2, 3.

(b) Denoting a correct answer by $C$ and a wrong answer by $W$, the elementary outcomes are given in the following table. For instance, $CWC$ denotes the elementary outcomes that the first and third questions were answered correctly and the second was wrong. In order to calculate the probabilities we note:

- For the first question, $P(C) = \frac{1}{4}$, $P(W) = \frac{3}{4}$.
- For the second question, $P(C) = \frac{1}{3}$, $P(W) = \frac{2}{3}$.
- For the third question, $P(C) = \frac{1}{2}$, $P(W) = \frac{1}{2}$.

Assuming the choices of answers to different questions to be independent, we multiply the $P(C)$ and $P(W)$ values according to the positions where $C$ and $W$ appear. For instance,

\[ P(CWC) = \frac{1}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{2}{24}. \]

<table>
<thead>
<tr>
<th>Elementary outcome</th>
<th>Probability</th>
<th>Value of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCC</td>
<td>$\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$</td>
<td>3</td>
</tr>
<tr>
<td>CCW</td>
<td>$\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$</td>
<td>2</td>
</tr>
<tr>
<td>CWC</td>
<td>$\frac{1}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{2}{24}$</td>
<td>2</td>
</tr>
<tr>
<td>WCC</td>
<td>$\frac{3}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{3}{24}$</td>
<td>2</td>
</tr>
<tr>
<td>CWW</td>
<td>$\frac{1}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{2}{24}$</td>
<td>1</td>
</tr>
<tr>
<td>WCW</td>
<td>$\frac{3}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{3}{24}$</td>
<td>1</td>
</tr>
<tr>
<td>WWC</td>
<td>$\frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{6}{24}$</td>
<td>1</td>
</tr>
<tr>
<td>WWW</td>
<td>$\frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{6}{24}$</td>
<td>0</td>
</tr>
</tbody>
</table>

The probability distribution of $X$ is
(c) \( P[X \geq 1] = f(1) + f(2) + f(3) = \frac{11}{24} + \frac{6}{24} + \frac{1}{24} = \frac{18}{24} = \frac{3}{4} \).

(d) The probability histogram of \( X \) is given by:

```
<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6/24</td>
</tr>
<tr>
<td>1</td>
<td>11/24</td>
</tr>
<tr>
<td>2</td>
<td>6/24</td>
</tr>
<tr>
<td>3</td>
<td>1/24</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
</tr>
</tbody>
</table>
```

5.40 (a) We calculate the values of \( Y = 8 - 2X \) corresponding to the values of \( X \) and list along with the probabilities \( f(x) \). The \( y \)-values and the corresponding probabilities are then listed in the table on the right which is the given \( Y \)-distribution.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 8 - 2x )</th>
<th>( f(x) )</th>
<th>( y )</th>
<th>( f(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>0.1</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.3</td>
<td>2</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.4</td>
<td>4</td>
<td>0.3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0.2</td>
<td>6</td>
<td>0.1</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( xf(x) )</th>
<th>( x^2f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.6</td>
<td>1.2</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>1.2</td>
<td>3.6</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>0.8</td>
<td>3.2</td>
</tr>
<tr>
<td>Total</td>
<td>2.7</td>
<td>8.1</td>
<td></td>
</tr>
</tbody>
</table>

\( \mu_X = 2.7 \)
\( \sigma_X^2 = 8.1 - (2.7)^2 = 0.81 \), and \( \sigma_X = \sqrt{0.81} = 0.9 \)
(c) | $y$ | $f(y)$ | $yf(y)$ | $y^2f(y)$  
--- | --- | --- | ---  
0 | 0.2 | 0 | 0  
2 | 0.4 | 0.8 | 1.6  
4 | 0.3 | 1.2 | 4.8  
6 | 0.1 | 0.6 | 3.6  
Total | | 2.6 | 10.0  

\[ \mu_r = 2.6 \]
\[ \sigma_r^2 = 10.0 - (2.6)^2 = 3.24, \quad \text{and} \quad \sigma_y = \sqrt{3.24} = 1.8 \]

(d) Since \( Y = 8 - 2X \), we should have
\[ \mu_r = 8 - 2\mu_x = 8 - 2(2.7) = 2.6 \]
\[ \sigma_r = 1 - 2\sigma_x = 2(0.9) = 1.8 \]

These results agree with the results of part (c).

5.78 (a) Identify S: an RY-offspring
Denote \( X = \) number of S's in 130 trials
Then, \( X \) is binomial with \( n = 130, \quad p = \frac{9}{9 + 3 + 3 + 1} = \frac{9}{16} \).
\[ E(X) = np = 130 \times \frac{9}{16} = 73.125 \]
\[ sd(X) = \sqrt{npq} = \sqrt{130 \times \frac{9}{16} \times \frac{7}{16}} = 5.656 \]

(b) Identify S: an WG-offspring
Denote \( Y = \) number of S's in 85 trials
Then, \( Y \) has the binomial distribution with \( n = 85, \quad p = \frac{1}{9 + 3 + 3 + 1} = \frac{1}{16} \).
\[ E(Y) = 85 \times \frac{1}{16} = 5.3125 \]
\[ sd(Y) = \sqrt{npq} = \sqrt{85 \times \frac{1}{16} \times \frac{15}{16}} = 2.2317. \]
(a) We calculate

\[
\begin{array}{|c|c|c|c|}
\hline
x & f(x) & xf(x) & x^2f(x) \\
\hline
0 & 0.13 & 0 & 0 \\
1 & 0.14 & 0.14 & 0.14 \\
2 & 0.43 & 0.86 & 1.72 \\
3 & 0.20 & 0.60 & 1.80 \\
4 & 0.10 & 0.40 & 1.60 \\
\hline
\end{array}
\]

so the mean \( \mu = 2 \) tickets and the standard deviation is \( \sqrt{5.26 - 2^2} = \sqrt{1.26} = 1.12 \) tickets.

(b) \( P(B) = P(X \geq 1) = f(1) + f(2) + f(3) + f(4) = 0.14 + 0.43 + 0.20 + 0.10 = 0.87 \)

and the intersection \( AB = [1 \leq X \leq 2] \) has probability \( P(AB) = f(1) + f(2) = 0.14 + 0.43 = 0.57 \). By the definition of conditional probability

\[
P[X \leq 2 | X \geq 1] = \frac{P(AB)}{P(B)} = \frac{0.57}{0.87} = 0.655
\]

(c) Under independence, we multiply the probabilities of 0 tickets for each day.

\[
P[0 \text{ tickets on exactly one of 5 days}] = 5(0.13)(0.87)^4 = 0.372
\]
5.120 \( P[1 \text{ success in } n \text{ trials}] = \binom{n}{1} pq^{n-1} = npq^{n-1} \)

\( P[0 \text{ successes in } n \text{ trials}] = q^n \)

One success is more probable than 0 successes if \( npq^{n-1} > q^n \) or

\[
\frac{q^n}{npq^{n-1}} = \frac{q}{p} = 0.85 = 5.67
\]

The smallest \( n \) (integer) that satisfies \( n > 5.67 \) is 6.

5.124 (a) The possible values of \( Y \) are 1, 2, ..., and the corresponding elementary outcomes are:

\[ [Y=1] = S, \quad [Y=2] = FS, \quad [Y=3] = FFS, \text{ etc.} \]

Thus,

\[ [Y = y] = \underbrace{FF...FS}_{y-1} \]

and its probability is

\[ f(y) = P[Y = y] = q \times p \times q \times p \times q \times p = q^{y-1} p. \]

(b) \( P[Y \leq 3] = f(1) + f(2) + f(3) \)

\[ = p + qp + q^2p = 0.5 + (0.5)(0.5) + (0.5)^2(0.5) = 0.875. \]

5.125 (a) \( P[X = 0] = f(0) = e^{-3} \frac{(3)^0}{0!} = e^{-3} = 0.05 \)

(b) \( P[X = 1] = f(1) = e^{-3} \frac{(3)^1}{1!} = e^{-3} \times 3 = 0.15. \)

5.126 (a) The center line is at \( \mu = 0.5 \), and the lower and upper control limits are

\[ 0.5 - 3 \sqrt{\frac{0.5 \times 0.5}{20}} = 0.165 \text{ and } 0.5 + 3 \sqrt{\frac{0.5 \times 0.5}{20}} = 0.835 \]
(b) The corresponding proportions 0.55, 0.4, 0.7, 0.5, 0.65, 0.6, 0.35, 0.7, 0.5 and 0.65 are graphed below.

(c) There are no days for which the proportion is out of control.

6.16 (a) \[ P[-0.75 < Z < 0.75] = P[Z < 0.75] - P[Z < -0.75] = 0.7734 - 0.2266 = 0.5468 \]
(b) \[ P[-1.09 < Z < 1.09] = P[Z < 1.09] - P[Z < -1.09] = 0.8621 - 0.1379 = 0.7242 \]
(c) \[ P[0.32 < Z < 2.65] = P[Z < 2.65] - P[Z < 0.32] = 0.9960 - 0.6255 = 0.3705 \]
(d) We find that
\[
\begin{align*}
P[Z < -0.74] &= 0.2296 \\
P[Z < -0.75] &= 0.2266 \\
difference &= 0.0030
\end{align*}
\]
Therefore,
\[
P[Z < -0.745] = 0.2266 + \frac{1}{2}(0.0030) = 0.2281 \text{ (rounded)}
\]
Also,
\[
\begin{align*}
P[Z < 1.25] &= 0.8944 \\
P[Z < 1.24] &= 0.8925 \\
difference &= 0.0019
\end{align*}
\]
So
\[
P[Z < 1.244] = 0.8925 + 0.4(0.0019) = 0.8933 \text{ (rounded)}
\]
Finally,
\[
P[-0.745 < Z < 1.244] = 0.8933 - 0.2281 = 0.6652
\]
6.18 (a) \( P[Z < -0.61] = 0.2709 \), so the \( z \)-value is \(-0.61\).

(b) We are to find the \( z \)-value for which the area to the left is \( 1-0.35 = 0.65 \).

From the normal table, we find that

\[
\begin{align*}
P[Z < 0.39] &= 0.6517 \\
P[Z < 0.38] &= 0.6480 \\
\text{difference} &= 0.0037
\end{align*}
\]

We need \( P[Z < z] = 0.65 \). Since \( 0.65 - 0.6480 = 0.002 \), and \( 0.39 - 0.38 = 0.01 \), the required \( z \)-value is

\[
Z = 0.38 + (0.01) \times \frac{0.002}{0.0037} = 0.38 + 0.0054 = 0.3854 \text{ (rounded)}
\]

(c) \( P[Z < z] = 0.5 + 0.38 = 0.88 \). From the normal table,

\[
\begin{align*}
P[Z < 1.18] &= 0.8810 \\
P[Z < 1.17] &= 0.8790 \\
\text{difference} &= 0.0020
\end{align*}
\]

We need \( P[Z < z] = 0.88 \). Since \( 0.88 - 0.8790 = 0.0010 \), and \( 1.18 - 1.17 = 0.01 \), the required \( z \)-value is

\[
Z = 1.17 + (0.01) \times \frac{0.0010}{0.0020} = 1.17 + 0.0050 = 1.175
\]

(d) Since \( P[Z < -1] = 0.1587 \), we require

\[
P[Z < z] = 0.730 + 0.1587 = 0.8887
\]

Scanning the normal table, we find \( P[Z < 1.22] = 0.8887 \) so \( z = 1.22 \) (no interpolation was needed).

(e) By the symmetry of the normal curve, we have

\[
2P[Z < -z] = 1 - P[-z < Z < z] = 1 - 0.7416 = 0.2584
\]

or

6.47 Let \( X = \) number of unemployed persons in a random sample of 300. Then, the distribution of \( X \) is binomial with \( n = 300 \) and \( p = 0.079 \), and a normal approximation is appropriate. Here

\[
np = 300 \times 0.079 = 23.7
\]

\[
\sqrt{npq} = \sqrt{300 \times 0.079 \times 0.921} = 4.672
\]

\[
Z = \frac{X - 23.7}{4.672}
\]
6.54 A binomial model is reasonable for \( X \) = number of infested trees out of 300.

(a) Since \( n = 300 \) and \( p = 0.2 \), the distribution of \( X \) is approximately normal with mean \( \mu = 300 \times 0.2 = 60 \), and \( sd = \sqrt{300 \times 0.2 \times 0.8} = 6.928 \)

\[
\Pr[49 \leq X \leq 71] = \Pr\left[ \frac{48.5 - 60}{6.928} < Z < \frac{71.5 - 60}{6.928} \right] = \Pr[-1.660 < Z < 1.660] = 0.9515 - 0.0485 = 0.9030
\]

(b) The observed count \( x = 72 \) is at the distance of \( 72 - 60 = 12 \) from the mean. We reason that the probability of getting this or a more extreme observation is

\[
\Pr[|X - 60| \geq 12] = 2 \Pr[Z > 1.732] \quad \text{(using calculations in part (a))}
\]

\[
= 2 \times 0.0416 = 0.0832
\]

The probability is 0.083, and this is small but not so small. The event of observing 72 or a more extreme result is not so rare as to contradict the hypothesis that that population proportion is 0.2.

6.70 (a) Denote the volume in one bottle by \( X \) which is a random variable having a normal distribution with mean 101.5 and standard deviation 1.6 ml. We want to find \( \Pr[X < 100] \). Since \( Z = \frac{X - 101.5}{1.6} \) is standard normal, we have

\[
\Pr[X < 100] = \Pr[Z < \frac{100 - 101.5}{1.6}] = \Pr[Z < -0.9375] = 0.1743
\]

(b) We first determine that the z-value 1.645 satisfies \( \Pr[Z \leq 1.645] = 0.9500 \) so \( \Pr[Z > 1.645] = 0.05 \). Consequently, \( v = 101.5 + (1.6) \times 1.645 = 104.13 \) milliliters.

7.6 The table below lists the 9 possible samples \((x_1, x_2)\), along with the corresponding values of \( \bar{x} \) and \( S^2 \). Since \( n = 2 \), the sample mean and sample variance for each member of the sample are calculated using these formulae:

\[
\bar{x} = \frac{x_1 + x_2}{2}, \quad S^2 = \frac{1}{n-1} \sum (x - \bar{x})^2 = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2
\]
For example, for \((x_1, x_2) = (2, 4)\), we have:
\[
\bar{x} = \frac{2 + 4}{2} = 3, \quad s^2 = (2 - 3)^2 + (4 - 3)^2 = 2
\]

<table>
<thead>
<tr>
<th>((x_1, x_2))</th>
<th>(0,0)</th>
<th>(0,2)</th>
<th>(0,4)</th>
<th>(2,0)</th>
<th>(2,2)</th>
<th>(2,4)</th>
<th>(4,0)</th>
<th>(4,2)</th>
<th>(4,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{x})</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(s^2)</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

(b) The 9 possible samples are equally likely, so each has a probability \(1/9\) of occurring. The sampling distributions of \(\bar{X}\) and \(S^2\) are obtained by listing the distinct values of \(\bar{X}\) and \(S^2\) along with the corresponding probabilities, as follows:

<table>
<thead>
<tr>
<th>(\bar{X})</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/9</td>
</tr>
<tr>
<td>1</td>
<td>2/9</td>
</tr>
<tr>
<td>2</td>
<td>3/9</td>
</tr>
<tr>
<td>3</td>
<td>2/9</td>
</tr>
<tr>
<td>4</td>
<td>1/9</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(S^2)</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3/9</td>
</tr>
<tr>
<td>2</td>
<td>4/9</td>
</tr>
<tr>
<td>8</td>
<td>2/9</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
</tr>
</tbody>
</table>

7.16 We first calculate the mean \(\mu\) and standard deviation \(\sigma\) of the population that corresponds to \(X\) taking the values 0, 2, and 4, each having the same probability of occurring (namely \(1/3\)).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
<th>(xf(x))</th>
<th>(x^2 f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1/3</td>
<td>2/3</td>
<td>4/3</td>
</tr>
<tr>
<td>4</td>
<td>1/3</td>
<td>4/3</td>
<td>16/3</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>6/3</td>
<td>20/3</td>
</tr>
</tbody>
</table>

Using the values in the table, we have the following:
\[
\mu = \sum xf(x) = \frac{8}{3} = 2
\]
\[
\sigma^2 = E(X^2) - \mu^2 = \sum x^2 f(x) - \mu^2 = \frac{20}{3} - 2^2 = \frac{8}{3}, \quad \text{so that } \sigma = \sqrt{\frac{8}{3}}
\]
For \( n = 2 \), we know that the mean and standard deviation of the sampling distribution of \( \bar{X} \) must be as follows:

\[
E(\bar{X}) = \mu = 2 \\
\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{\frac{2}{2}} = \sqrt{\frac{1}{2}}
\]

We verify these by actually calculating the distribution of \( \bar{X} \):

<table>
<thead>
<tr>
<th>( \bar{X} )</th>
<th>( f(\bar{X}) )</th>
<th>( \bar{x} f(\bar{X}) )</th>
<th>( \bar{x}^2 f(\bar{X}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2/9</td>
<td>2/9</td>
<td>2/9</td>
</tr>
<tr>
<td>2</td>
<td>3/9</td>
<td>6/9</td>
<td>12/9</td>
</tr>
<tr>
<td>3</td>
<td>2/9</td>
<td>6/9</td>
<td>18/9</td>
</tr>
<tr>
<td>4</td>
<td>1/9</td>
<td>4/9</td>
<td>16/9</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>18/9</td>
<td>48/9</td>
</tr>
</tbody>
</table>

Using the values in the table, we have the following (which do indeed confirm the above assertion):

\[
E(\bar{X}) = \sum \bar{x} f(\bar{X}) = \frac{18}{9} = 2 \\
\text{Var}(\bar{X}) = E(\bar{X}^2) - (E(\bar{X}))^2 = \sum \bar{x}^2 f(\bar{X}) - (E(\bar{X}))^2 = \frac{48}{9} - 2^2 = \frac{12}{9} = \frac{4}{3} \\
\text{sd}(\bar{X}) = \sqrt{\frac{4}{3}}
\]

7.25 The population of fry has mean \( \mu = 3.4 \) and standard deviation \( \sigma = 0.8 \), so that

\[
E(\bar{X}) = \mu = 3.4 \quad \text{and} \quad \text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{0.8}}{\sqrt{50}} = 0.1333
\]

and the standardized variable is \( Z = \frac{\bar{X} - 3.4}{0.1333} \).

(a) \( P(\bar{X} < 3.2) = P(Z < \frac{3.2 - 3.4}{0.1333}) = P(Z < -1.5) = 0.0668 \)

(b) Those caught in the net may be slower, less active fish, or even the less healthy ones. Consequently, they may tend to be on the smaller side of the distribution.