Online Supplement to
"Testing for stationarity of functional time series in the frequency domain"*

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Abstract

This supplement contains additional technical material necessary to complete the proofs of theorems of the main paper Aue & van Delft (2016). Section S1 contains the proofs of several auxiliary lemmas stated in Appendix A of the main paper. Section S2 contains results on finding bounds for the denominator of the test statistic. Sections S3 and S4 deal with convergence of the finite-dimensional distributions and tightness. Section S5 establishes the asymptotic covariance structure of the test under the alternative of local stationarity.

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S1 Properties of functional cumulants

Proof of Lemma A.2. By linearity of the cumulant operation, consecutively taking differences leads, by equation (4.4) of the main paper and the triangle inequality, to

$$\Big\|\mathrm{cum}\big(X_{t_1}^{(T)},\dots,X_{t_k}^{(T)}\big)-\mathrm{cum}\big(X_{t_1}^{(t_1/T)},\dots,X_{t_k}^{(t_k/T)}\big)\Big\|_2=\Big\|\sum_{j=1}^{k}\mathrm{cum}\big(X_{t_1}^{(T)},\dots,Y_{t_j}^{(T)},\dots,X_{t_k}^{(T)}\big)\Big\|_2$$
$$\leqslant K\frac{k}{T}\|\kappa_{k;t_1-t_k,\dots,t_{k-1}-t_k}\|_2,$$

using part (i) of Assumption 4.3. By (4.4),

$$X_{t_j}^{(t_j/T)}-X_{t_j}^{(t_k/T)}=\frac{(t_j-t_k)}{T}Y_{t_j}^{(t_j/T,t_k/T)}. \tag{S1.1}$$

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Similarly,

$$\left\| \operatorname{cum}\big(X_{t_1}^{(t_1/T)}, \dots, X_{t_k}^{(t_k/T)}\big) - c_{t_1/T; t_1-t_k, \dots, t_{k-1}-t_k} \right\|_2$$
$$\leqslant \sum_{j=1}^{k-1} \frac{|t_j - t_k|}{T} \|\kappa_{k; t_1-t_k, \dots, t_{k-1}-t_k}\|_2,$$

which follows from part (iii) of Assumption 4.3 Minkowski's inequality then implies the lemma. □

Proof of Lemma A.3. Using Lemma A.2, it is straightforward to show that $(X_t^T : t \leqslant T, T \in \mathbb{N})$ uniquely determines the time-varying spectral density operator, that is,

$$\int_{-\pi}^{\pi} \|\mathcal{F}_{u,\omega}^{(T)} - \mathcal{F}_{u,\omega}\|_2^2 \, d\omega = o(1) \qquad (T \to \infty). \tag{S1.2}$$

Existence of the derivatives follow from the dominated convergence theorem and the product rule for differentiation in Banach spaces (Nelson, 1969). □

Proof of Lemma A.4. The first line of (A.4) follows on replacing the cumulants $\operatorname{cum}(X_{t_1}^{(T)}, \dots, X_{t_{k-1}}^{(T)}, X_{t_k}^{(T)})$ with $c_{t_k/T; t_1-t_k, \dots, t_{k-1}-t_k}$ and Lemma A.2. The second line because the discretization of the integral is an operation of order $O(T^{-2})$.

By Assumption 4.3 (iv), the kernel of $\frac{\partial}{\partial u} \mathcal{F}_{u;\omega_1, \dots, \omega_{k-1}}$ satisfies

$$\left\| \sup_u \frac{\partial}{\partial u} f_{u;\omega_1, \dots, \omega_{k-1}} \right\|_2 \leqslant \frac{1}{(2\pi)^{k-1}} \sum_{t_1, \dots, t_k} \|\kappa_{k; t_1-t_k, \dots, t_{k-1}-t_k}\|_2 < \infty.$$

The dominated convergence theorem therefore yields

$$\sup_{u, \omega_1, \dots, \omega_{k-1}} \left\| \frac{\partial}{\partial u} f_{u, \omega_1, \dots, \omega_{k-1}} \right\|_2 < \infty. \tag{S1.3}$$

Finally, integration by parts for a periodic function in $L^2([0,1]^k)$ with existing n-th directional derivative in u, yields

$$\begin{aligned}
&\|\tilde{f}_{s;\omega_{j_1}, \dots, \omega_{j_{k-1}}}\|_2^2 \\
&= \int_{[0,1]^k} \left| \left[\frac{\frac{\partial^{n-1}}{\partial u^{n-1}} f_{u;\omega_{j_1}, \dots, \omega_{j_{k-1}}}(\boldsymbol{\tau})}{(-\mathrm{i}2\pi s)^{n-1}} e^{-\mathrm{i}s2\pi u} \right]_0^1 - \int_0^1 \frac{e^{-\mathrm{i}s2\pi u}}{(-\mathrm{i}2\pi s)^n} \frac{\partial^n}{\partial u^n} f_{u;\omega_{j_1}, \dots, \omega_{j_{k-1}}}(\boldsymbol{\tau}) du \right|^2 d\boldsymbol{\tau} \\
&= \int_{[0,1]^{k+2}} \frac{1}{(2\pi s)^{2n}} e^{\mathrm{i}2\pi s(u-v)} \frac{\partial^2}{\partial u^2} f_{u;\omega_{j_1}, \dots, \omega_{j_{k-1}}}(\boldsymbol{\tau}) \frac{\partial^2}{\partial v^2} f_{v;\omega_{j_1}, \dots, \omega_{j_{k-1}}}(\boldsymbol{\tau}) d\boldsymbol{\tau} du dv \\
&\leqslant \frac{1}{(2\pi s)^{2n}} \int_{[0,1]^{k+2}} \left| \frac{\partial^2}{\partial u^2} f_{u;\omega_{j_1}, \dots, \omega_{j_{k-1}}}(\boldsymbol{\tau}) \frac{\partial^2}{\partial v^2} f_{v;\omega_{j_1}, \dots, \omega_{j_{k-1}}}(\boldsymbol{\tau}) \right| d\boldsymbol{\tau} du dv \\
&\leqslant \frac{1}{(2\pi s)^{2n}} \int_{[0,1]^2} \left\| \frac{\partial^2}{\partial u^2} f_{u;\omega_{j_1}, \dots, \omega_{j_{k-1}}} \right\|_2 \left\| \frac{\partial^2}{\partial v^2} f_{v;\omega_{j_1}, \dots, \omega_{j_{k-1}}} \right\|_2 du dv \\
&\leqslant \frac{1}{(2\pi s)^{2n}} \left(\sup_u \left\| \frac{\partial^2}{\partial u^2} f_{u;\omega_{j_1}, \dots, \omega_{j_{k-1}}} \right\|_2 \right)^2 < \infty,
\end{aligned}$$

where the Cauchy–Schwarz inequality was applied in the second-to-last equality. The interchange of integrals is justified by Fubini's theorem. Thus,

$$\sup_{\omega_1,..,\omega_{k-1}} \|\tilde{f}_{s;\omega_{j_1},\dots,\omega_{j_{k-1}}}\|_2 \leqslant \frac{1}{(2\pi)^{2n}} \sup_{u,\omega_1,\dots,\omega_n} \left\| \frac{\partial^n}{\partial u^n} f_{u;\omega_{j_1},\dots,\omega_{j_{k-1}}} \right\|_2 |s|^{-n}. \tag{S1.4}$$

and (A.6) follows from Assumption 4.3 (iv). □

Proof of Corollary A.1. For completeness, we elaborate on part (ii) of the corollary. Note that

$$|||\tilde{\mathcal{F}}_{0:\omega}|||_2 \leqslant \sup_{\omega,u} |||\mathcal{F}_{u,\omega}|||_2 < \sum_h \|\kappa_{2,h}\|_2 < \infty.$$

The p-harmonic series for $p = 2$ then yields

$$\sup_{\omega} \sum_{s \in \mathbb{Z}} |||\tilde{\mathcal{F}}_{s;\omega}|||_2 \leqslant \sum_h \|\kappa_{2,h}\|_2 \left(1 + \frac{1}{(2\pi)^4} \frac{\pi^2}{3}\right) < \infty, \tag{S1.5}$$

where the constant $(2\pi)^{-4}$ follows from (S1.4). □

S2 Error bound for the denominator of the test statistic

A bound needs to be obtained on the error resulting from the replacement of the unknown spectral operators with consistent estimators. It will be sufficient to consider a bound on

$$\sqrt{T}|\gamma_h^{(T)}(l_1, l_2) - \hat{\gamma}_h^{(T)}(l_1, l_2)|$$

for all $l_1, l_2 \in \mathbb{N}$, where $\gamma^{(T)}$ is defined in equation (3.1) of the main paper. Consider the function $g(x) = x^{-1/2}$, $x > 0$, and notice that

$$\gamma_h^{(T)}(l_1, l_2) - \hat{\gamma}_h^{(T)}(l_1, l_2) = \frac{1}{T} \sum_{j=1}^{T} D_{\omega_j}^{(l_1)} D_{-\omega_{j+h}}^{(l_2)} \Big[g(\mathcal{F}_{\omega_j}^{(l_1,l_1)} \mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)}) - g(\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} \hat{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)}) \Big].$$

Given the assumption $\inf_\omega \langle \mathcal{F}_\omega(\psi), \psi \rangle > 0$ for all $\psi \in H$ is satisfied, continuity of the inner product implies that for fixed l_1, l_2 the mean value theorem may be applied to find

$$\begin{aligned} &\gamma_h^{(T)}(l_1, l_2) - \hat{\gamma}_h^{(T)}(l_1, l_2) \\ &\qquad = \frac{1}{T} \sum_{j=1}^{T} D_{\omega_j}^{(l_1)} D_{-\omega_{j+h}}^{(l_2)} \left[\frac{\partial g(x)}{\partial x} \Big|_{x=\breve{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} \breve{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)}} \big(\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} \hat{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)} - \mathcal{F}_{\omega_j}^{(l_1,l_1)} \mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)} \big) \right], \end{aligned}$$

where $\breve{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} \breve{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)}$ lies in between $\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} \hat{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)}$ and $\mathcal{F}_{\omega_j}^{(l_1,l_1)} \mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)}$. Because of uniform convergence of $\hat{\mathcal{F}}_{\omega_j} \in S_2(H)$ with respect to ω, it follows that

$$\sqrt{T}|\gamma_h^{(T)}(l_1, l_2) - \hat{\gamma}_h^{(T)}(l_1, l_2)| = O_p(1) \left| \frac{1}{T} \sum_{j=1}^{T} \frac{D_{\omega_j}^{(l_1)} D_{-\omega_{j+h}}^{(l_2)}}{(\mathcal{F}_{\omega_j}^{(l_1,l_1)} \mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)})^{3/2}} \big(\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} \hat{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)} - \mathcal{F}_{\omega_j}^{(l_1,l_1)} \mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)} \big) \right|. \tag{S2.1}$$

With these preliminary results at hand, the first main theorems can be verified.

Proof of Theorems 4.1 and 4.5. In order to prove both theorems, partition (S2.1) as follows

$$J_1(l,l_2) = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} \frac{D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}} - \mathbb{E}[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}]}{(\mathcal{F}^{(l_1,l_1)}_{\omega_j}\mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}})^{3/2}} \hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j} \big(\hat{\mathcal{F}}^{(l_2,l_2)}_{\omega_{j+h}} - \mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}}\big), \tag{S2.2}$$

$$J_2(l,l_2) = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} \frac{D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}} - \mathbb{E}[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}]}{(\mathcal{F}^{(l_1,l_1)}_{\omega_j}\mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}})^{3/2}} \mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}} \big(\hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j} - \mathcal{F}^{(l_1,l_1)}_{\omega_j}\big), \tag{S2.3}$$

$$J_3(l,l_2) = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} \frac{\mathbb{E}[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}]}{(\mathcal{F}^{(l_1,l_1)}_{\omega_j}\mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}})^{3/2}} \big(\hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j}\hat{\mathcal{F}}^{(l_2,l_2)}_{\omega_{j+h}} - \mathbb{E}[\hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j}\hat{\mathcal{F}}^{(l_2,l_2)}_{\omega_{j+h}}]\big), \tag{S2.4}$$

$$J_4(l,l_2) = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} \frac{\mathbb{E}[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}]}{(\mathcal{F}^{(l_1,l_1)}_{\omega_j}\mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}})^{3/2}} \big(\mathbb{E}[\hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j}\hat{\mathcal{F}}^{(l_2,l_2)}_{\omega_{j+h}}] - \mathcal{F}^{(l_1,l_1)}_{\omega_j}\mathcal{F}^{(l_2,l_2)}_{\omega_{j+h}}\big). \tag{S2.5}$$

The proof of both theorems are based on Corollary S2.1 and Lemmas S2.1-S2.4 below. These results together with an application of the Cauchy–Schwarz inequality yield

(i) $|J_1| = O_p\Big(\frac{1}{\sqrt{bT}} + b^2\Big)$,

(ii) $|J_2| = \begin{cases} O_p\Big(\frac{1}{\sqrt{bT}} + \frac{b^2}{\sqrt{T}}\Big) & \text{under Assumption 4.1,} \\ O_p\Big(\frac{1}{\sqrt{bT}} + b^2\Big) & \text{under Assumption 4.3,} \end{cases}$

(iii) $|J_3| = \begin{cases} O_p\Big(\frac{1}{\sqrt{bT}}\Big) & \text{under Assumption 4.1,} \\ O_p\Big(\frac{1}{\sqrt{bT}}\Big) & \text{under Assumption 4.3,} \end{cases}$

(iv) $|J_4| = \begin{cases} O\Big(b^2 + \frac{1}{bT}\Big) & \text{under Assumption 4.1.} \\ O\Big(\sqrt{T}b^2 + \frac{1}{b\sqrt{T}}\Big) & \text{under Assumption 4.3.} \end{cases}$

Minkowski's Inequality then gives the result. □

Corollary S2.1. *Under Assumption 4.1*

$$\mathbb{E}\big[\| \hat{\mathcal{F}}^{(T)}_\omega - \mathcal{F}_\omega \|_2^2\big] = O\Big(\frac{1}{bT} + b^4\Big) \qquad (T \to \infty), \tag{S2.6}$$

while, under Assumption 4.3,

$$\mathbb{E}\big[\| \hat{\mathcal{F}}^{(T)}_\omega - G_\omega \|_2^2\big] = O\Big(\frac{1}{bT} + b^4\Big) \qquad (T \to \infty). \tag{S2.7}$$

Proof. See Panaretos & Tavakoli (2013, Theorem 3.6) and Theorem 4.4, respectively. □

Lemma S2.1. *Let* $(g_j^{(l_1,l_2)} : j \in \mathbb{Z})$ *be a bounded sequence in* $\mathbb{C}$ *for all* $l_1, l_2 \in \mathbb{N}$ *such that* $\inf_j g_j^{(l_1,l_2)} > 0$. *Under Assumption 4.1 and under Assumption 4.3,*

$$\mathbb{E}\Big[\Big|\frac{1}{\sqrt{T}}\sum_{j=1}^{T} g_j^{(l_1,l_2)}\big(D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)} - \mathbb{E}\big[D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)}\big]\big)\hat{\mathfrak{F}}_{\omega_j}^{(l_1,l_1)}\Big|^2\Big] = O(1).$$

Proof. Observe that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\frac{1}{\sqrt{T}}\sum_{j=1}^{T} g_j^{(l_1,l_2)}\big(D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)} - \mathbb{E}\big[D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)}\big]\big)\hat{\mathfrak{F}}_{\omega_j}^{(l_1,l_1)}\Big|^2\Big] \\
&= \mathbb{E}\Big[\Big|\frac{1}{\sqrt{T}}\sum_{j=1}^{T} g_j^{(l_1,l_2)}\big(D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)} - E\big[D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)}\big]\big)\frac{2\pi}{bT}\sum_{j'=1}^{T} K\Big(\frac{\omega_{j'}}{b}\Big)D_{\omega_{j-j'}}^{(l_1)}D_{\omega_{j'-j}}^{(l_1)}\Big|^2\Big] \\
&= \frac{1}{T}\Big(\frac{2\pi}{bT}\Big)^2\sum_{j_1,j_2=1}^{T} g_{j_1,j_2}^{(l_1,l_2)}\sum_{j_1',j_2'=1}^{T} K\Big(\frac{\omega_{j_1'}}{b}\Big)K\Big(\frac{\omega_{j_2'}}{b}\Big)\mathbb{E}\Big[\big\{\big(D_{\omega_{j_1}}^{(l_1)}D_{-\omega_{j_1+h}}^{(l_2)} - \mathbb{E}\big[D_{\omega_{j_1}}^{(l_1)}D_{-\omega_{j_1+h}}^{(l_2)}\big]\big) \\
&\qquad \times D_{\omega_{j_1-j_1'}}^{(l_1)}D_{\omega_{j_1'-j_1}}^{(l_1)}\big\}\big\{\big(D_{-\omega_{j_2}}^{(l_1)}D_{\omega_{j_2+h}}^{(l_2)} - \mathbb{E}\big[D_{-\omega_{j_2}}^{(l_1)}D_{\omega_{j_2+h}}^{(l_2)}\big]\big)D_{\omega_{j_2'-j_2}}^{(l_1)}D_{\omega_{j_2-j_2'}}^{(l_1)}\big\}\Big].
\end{aligned}$$

Expanding the expectation in terms of cumulants, one obtains the structure

$$\begin{aligned}
&\mathbb{E}[(X-\mathbb{E}X)Y][(W-EW)Z] \\
&\quad = E[XYWZ] - E[W]E[XYZ] - E[YWZ[E[X] + E[W]E[X]E[YZ] \\
&\quad = \mathrm{cum}(X,Y,W,Z) + \mathrm{cum}(X,Y,W)\mathrm{cum}(Z) + \mathrm{cum}(X,W,Z)\mathrm{cum}(Y) \\
&\qquad + \mathrm{cum}(X,Y)\mathrm{cum}(W,Z) + \mathrm{cum}(X,W)\mathrm{cum}(Y,Z) + \mathrm{cum}(X,Z)\mathrm{cum}(Y,W),
\end{aligned}$$

where X, Y, W, Z are products of random elements of H. Hence, by the product theorem for cumulants, only those products of cumulants have to be considered that lead to indecomposable partitions of the matrix below or of any sub-matrix (with the same column structure) with the exception that (Y) or (Z) is allowed to be decomposable but not within the same partition.

$$\begin{array}{lll}
(X) & D_{\omega_{j_1}}^{(l_1)} & D_{-\omega_{j_1+h}}^{(l_2)} \\
(Y) & D_{\omega_{j_1-j_1'}}^{(l_1)} & D_{\omega_{j_1'-j_1}}^{(l_1)} \\
(W) & D_{-\omega_{j_2}}^{(l_1)} & D_{\omega_{j_2+h}}^{(l_2)} \\
(Z) & D_{\omega_{j_2'-j_2}}^{(l_1)} & D_{\omega_{j_2-j_2'}}^{(l_1)}
\end{array} \tag{S2.8}$$

That is, the order of the error belonging to those partitions has to be investigated for which the cumulant terms that contain an element of the sets $\{D_{\omega_{j_1}}^{(l_1)}, D_{-\omega_{j_1+h}}^{(l_2)}\}$ and of $\{D_{-\omega_{j_2}}^{(l_1)} D_{\omega_{j_2+h}}^{(l_2)}\}$ also contain at least one element from another set. Because the process has zero-mean, it suffices consider partitions for which $m_i \geqslant 2$. By Lemma A.1 of the main paper, a cumulant of order k upscaled by order T will be of order $O(T^{-k/2+1})$ under H_0 and $O(T^{-k/2+2})$ under the alternative. This directly implies that only terms of the following form have to be investigated:

$$\frac{1}{T}\left(\frac{2\pi}{bT}\right)^2 \sum_{j_1,j_2=1}^{T} \sum_{j_1',j_2'=1}^{T} K\left(\frac{\omega_{j_1'}}{b}\right) K\left(\frac{\omega_{j_2'}}{b}\right) \mathrm{cum}_4 \mathrm{cum}_2 \mathrm{cum}_2, \tag{S2.9}$$

$$\frac{1}{T}\left(\frac{2\pi}{bT}\right)^2 \sum_{j_1,j_2=1}^{T} \sum_{j_1',j_2'=1}^{T} K\left(\frac{\omega_{j_1'}}{b}\right) K\left(\frac{\omega_{j_2'}}{b}\right) \mathrm{cum}_3 \mathrm{cum}_3 \mathrm{cum}_2, \tag{S2.10}$$

$$\frac{1}{T}\left(\frac{2\pi}{bT}\right)^2 \sum_{j_1,j_2=1}^{T} \sum_{j_1',j_2'=1}^{T} K\left(\frac{\omega_{j_1'}}{b}\right) K\left(\frac{\omega_{j_2'}}{b}\right) \mathrm{cum}_2 \mathrm{cum}_2 \mathrm{cum}_2 \mathrm{cum}_2. \tag{S2.11}$$

However, for a fixed partition $P = \{P_1, \ldots, P_M\}$,

$$T \prod_{j=1}^{M} O\left(\frac{1}{T^{m_j/2-1}}\right),$$

from which it is also clear that (S2.9) and (S2.10) are at most going to be of order $O(1)$ under the alternative and of lower order under H_0. The term (S2.11) could possibly of order $O(T)$. Further analysis can therefore be restricted to partitions of this form. As mentioned above, partitions in which a term contains an element of the sets $\{D^{(l_1)}_{\omega_{j_1}}, D^{(l_2)}_{-\omega_{j_1+h}}\}$ and of $\{D^{(l_1)}_{-\omega_{j_2}} D^{(l_2)}_{\omega_{j_2+h}}\}$ must contain at least one element from another set. It follows that partitions in which $m_i = 2$ for all $i \in \{1, \ldots, M\}$, without any restrictions on the summations, are decomposable. We find the term of highest order is thus of the type

$$\frac{1}{T} \sum_{j_1,j_2=1}^{T} g^{(l_1,l_2)}_{j_1,j_2} \left(\frac{2\pi}{bT}\right)^2 \sum_{j_1',j_2'=1}^{T} K\left(\frac{\omega_{j_1'}}{b}\right) K\left(\frac{\omega_{j_2'}}{b}\right) \mathrm{cum}\big(D^{(l_2)}_{-\omega_{j_1+h}}, D^{(l_2)}_{\omega_{j_2+h}}\big) \mathrm{cum}\big(D^{(l_1)}_{-\omega_{j_2}}, D^{(l_1)}_{\omega_{j_1}}\big)$$
$$\times\, \mathrm{cum}\big(D^{(l_1)}_{\omega_{j_1-j_1'}} D^{(l_1)}_{\omega_{j_1-j_1'}}\big) \mathrm{cum}\big(D^{(l_1)}_{\omega_{j_2'-j_2}} D^{(l_1)}_{\omega_{j_2-j_2'}}\big) = \sup_j g^{2(l_1,l_2)}_j O(TT^{-1}) = O(1),$$

under the null of stationarity and under the alternative, where the error is uniform in ω. The bound in case of the alternative follows from Corollary A.1 of the main paper. The result follows since, by the positive definiteness of the spectral density operators, $\sup_j |g^{(l_1,l_2)}_j| < \infty$ for all $l_1, l_2 \in \mathbb{N}$. □

Lemma S2.2. *Let* $(g^{(l_1,l_2)}_j : j \in \mathbb{Z})$ *be a bounded sequence in* $\mathbb{C}$ *for all* $l_1, l_2 \in \mathbb{N}$ *such that* $\inf_{l_1,l_2} g^{(l_1,l_2)}_j > 0$. *Then,*

$$E\left[\left|\frac{1}{\sqrt{T}} \sum_{j=1}^{T} g^{(l_1,l_2)}_j \big(D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}} - \mathbb{E}\big[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}\big]\big)\right|^2\right] = \begin{cases} O\left(\frac{1}{T}\right) & \textit{under } H_0. \\ O(1) & \textit{under } H_1. \end{cases}$$

Proof. Notice that

$$E\left[\left|\frac{1}{\sqrt{T}} \sum_{j=1}^{T} g^{(l_1,l_2)}_j \big(D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}} - \mathbb{E}\big[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}\big]\big)\right|^2\right]$$
$$= \frac{1}{T} \sum_{j_1,j_2=1}^{T} \left(g^{(l_1,l_2)}_{j_1,j_2} \mathbb{E}\big[D^{(l_1)}_{\omega_{j_1}} D^{(l_2)}_{-\omega_{j_1+h}} D^{(l_1)}_{-\omega_{j_2}} D^{(l_2)}_{\omega_{j_2+h}}\big] - \mathbb{E}\big[D^{(l_1)}_{\omega_{j_1}} D^{(l_2)}_{-\omega_{j_1+h}}\big] \mathbb{E}\big[D^{(l_1)}_{-\omega_{j_2}} D^{(l_2)}_{\omega_{j_2+h}}\big] \right)$$

$$= \frac{1}{T} \sum_{j_1,j_2=1}^{T} \Big(g^{(l_1,l_2)}_{j_1,j_2} \mathrm{cum}\big(D^{(l_1)}_{\omega_{j_1}}, D^{(l_2)}_{-\omega_{j_1+h}}, D^{(l_1)}_{-\omega_{j_2}} D^{(l_2)}_{\omega_{j_2+h}}\big) +$$
$$\mathrm{cum}\big(D^{(l_1)}_{\omega_{j_1}}, D^{(l_1)}_{-\omega_{j_2}}\big)\mathrm{cum}\big(D^{(l_2)}_{-\omega_{j_1+h}}, D^{(l_2)}_{\omega_{j_2+h}}\big) + \mathrm{cum}\big(D^{(l_1)}_{\omega_{j_1}}, D^{(l_2)}_{\omega_{j_2+h}}\big)\mathrm{cum}\big(D^{(l_2)}_{-\omega_{j_1+h}}, D^{(l_1)}_{-\omega_{j_2}}\big)\Big).$$

Under the null, this is therefore of the order $O(T/T^2 + 1/T) = O(1/T)$, where is uniform over ω. Under the alternative, by Corrolary A.1 of the main paper, the last term can be estimated by

$$\frac{1}{T} \sum_{j_1,j_2=1}^{T} g^{(l_1,l_2)}_{j_1,j_2} \bigg(\frac{1}{T} G^{(l_1\, l_2, l_1\, l_2)}_{\omega_{j_1}, -\omega_{j_1+h}, -\omega_{j_2}} + \tilde{\mathfrak{F}}^{(l_1,l_1)}_{j_1-j_2;\omega_{j_1}} \tilde{\mathfrak{F}}^{(l_2,l_2)}_{-j_1+j_2;-\omega_{j_1+h}}$$
$$+ \tilde{\mathfrak{F}}^{(l_1,l_2)}_{j_1+j_2+h;\omega_{j_1}} \tilde{\mathfrak{F}}^{(l_2,l_1)}_{-j_1-j_2-h;-\omega_{j_1+h}} + O\Big(\frac{1}{T}\Big)\bigg)$$
$$\leqslant \sup_j |g^{(l_1,l_2)}_j|^2 \frac{1}{T} \sum_{j_1,j_2=1}^{T} \bigg(\frac{1}{T} G^{(l_1\, l_2, l_1\, l_2)}_{\omega_{j_1}, -\omega_{j_1+h}, -\omega_{j_2}} + \tilde{\mathfrak{F}}^{(l_1,l_1)}_{j_1-j_2;\omega_{j_1}} \tilde{\mathfrak{F}}^{(l_2,l_2)}_{-j_1+j_2;-\omega_{j_1+h}}$$
$$+ \tilde{\mathfrak{F}}^{(l_1,l_2)}_{j_1+j_2+h;\omega_{j_1}} \tilde{\mathfrak{F}}^{(l_2,l_1)}_{-j_1-j_2-h;-\omega_{j_1+h}} + O\Big(\frac{1}{T}\Big)\bigg)$$
$$\leqslant \sup_j |g^{(l_1,l_2)}_j|^2 \bigg(\sum_{t_1,t_2,t_3} \|\kappa_{4;t_1,t_2,t_3}\|_2 \|\psi_{l_1}\|_2^2 |\psi_{l_2}\|_2^2 + C \sum_t \|\kappa_{2;t}\|_2 \bigg) = O(1),$$

for some constant C. □

Lemma S2.3. *Let* $(g^{(l_1,l_2)}_j : j \in \mathbb{Z})$ *be a bounded sequence in* $\mathbb{C}$ *for all* $l_1, l_2 \in \mathbb{N}$ *such that* $\inf_{l_1,l_2} g^{(l_1,l_2)}_j > 0$. *Then,*

$$\mathbb{E}\bigg[\bigg|\frac{1}{\sqrt{T}} \sum_{j=1}^{T} g^{(l_1,l_2)}_j \mathbb{E}\big[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}\big] \Big(\hat{\mathfrak{F}}^{(l_1,l_1)}_{\omega_j} \hat{\mathfrak{F}}^{(l_2,l_2)}_{\omega_{j+h}} - \mathbb{E}\big[\hat{\mathfrak{F}}^{(l_1,l_1)}_{\omega_j} \hat{\mathfrak{F}}^{(l_2,l_2)}_{\omega_{j+h}}\big]\Big)\bigg|\bigg]$$
$$= \begin{cases} O\Big(\dfrac{1}{\sqrt{b}T}\Big) & \textit{under Assumption 4.1.} \\ O\Big(\dfrac{1}{\sqrt{bT}}\Big) & \textit{under Assumption 4.3.} \end{cases}$$

Proof. Observe first that by the Cauchy–Schwarz inequality,

$$\mathbb{E}[|J_3(l_1,l_2)|] \leqslant \sup_j \big|g^{(l_1,l_2)}_j \mathbb{E}\big[D^{(l_1)}_{\omega_j} D^{(l_2)}_{-\omega_{j+h}}\big]\big|$$
$$\times \bigg(\mathbb{E}\bigg[\bigg|\frac{1}{\sqrt{T}} \sum_{j=1}^{T} \big(\hat{\mathfrak{F}}^{(l_1,l_1)}_{\omega_j} \hat{\mathfrak{F}}^{(l_2,l_2)}_{\omega_{j+h}} - \mathbb{E}\big[\hat{\mathfrak{F}}^{(l_1,l_1)}_{\omega_j} \hat{\mathfrak{F}}^{(l_2,l_2)}_{\omega_{j+h}}\big]\big)\bigg|^2\bigg]\bigg)^{1/2},$$

which follows because the term over which the supremum is taken is deterministic. In particular, it is of order $O(T^{-1})$ under the null and $O(h^{-2})$ under the alternative. To find a bound on

$$\mathbb{E}\bigg[\bigg|\frac{1}{\sqrt{T}} \sum_{j=1}^{T} \big(\hat{\mathfrak{F}}^{(l_1,l_1)}_{\omega_j} \hat{\mathfrak{F}}^{(l_2,l_2)}_{\omega_{j+h}} - \mathbb{E}\big[\hat{\mathfrak{F}}^{(l_1,l_1)}_{\omega_j} \hat{\mathfrak{F}}^{(l_2,l_2)}_{\omega_{j+h}}\big]\big)\bigg|^2\bigg],$$

we proceed similarly as in the proof of Lemma S2.1. Observe that,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\frac{1}{\sqrt{T}}\sum_{j=1}^{T}\big(\hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j}\hat{\mathcal{F}}^{(l_2,l_2)}_{\omega_{j+h}}-\mathbb{E}\hat{\mathcal{F}}^{(l_1,l_1)}_{\omega_j}\hat{\mathcal{F}}^{(l_2,l_2)}_{\omega_{j+h}}\big)\Big|^2\Big]\\
=&\frac{1}{T}\Big(\frac{2\pi}{bT}\Big)^2\sum_{j_1,j_2=1}^{T}\sum_{j'_1,j'_2,j'_3,j'_4=1}^{T}\prod_{i=1}^{4}K\Big(\frac{\omega_{j'_i}}{b}\Big)\\
&\times\mathbb{E}\Big[\big(D^{(l_1)}_{\omega_{j_1-j'_1}}D^{(l_1)}_{\omega_{j'_1-j_1}}D^{(l_2)}_{\omega_{j_1+h-j'_2}}D^{(l_2)}_{\omega_{j'_2-j_1-h}}-E[D^{(l_1)}_{\omega_{j_1-j'_1}}D^{(l_1)}_{\omega_{j'_1-j_1}}D^{(l_2)}_{\omega_{j_1+h-j'_2}}D^{(l_2)}_{\omega_{j'_2-j_1-h}}]\big)\\
&\times\big(D^{(l_1)}_{\omega_{j'_3-j_2}}D^{(l_1)}_{\omega_{j_2-j'_3}}D^{(l_2)}_{\omega_{j'_4-j_2-h}}D^{(l_2)}_{\omega_{j_2+h-j'_4}}-\mathbb{E}[D^{(l_1)}_{\omega_{j'_3-j_2}}D^{(l_1)}_{\omega_{j_2-j'_3}}D^{(l_2)}_{\omega_{j'_4-j_2-h}}D^{(l_2)}_{\omega_{j_2+h-j'_4}}]\big)\Big].
\end{aligned}
$$

Write

$$
\mathbb{E}[(X-\mathbb{E}X)][(Y-EY)]=\text{cum}(X,Y)-\text{cum}(X)\text{cum}(Y)
$$

for products X, W of random elements of H. When expanding this in terms of cumulants, we only have to consider those products of cumulants that lead to indecomposable partitions of the rows of the matrix below

$$
\begin{array}{lllll}
(X) & D^{(l_1)}_{\omega_{j_1-j'_1}} & D^{(l_1)}_{\omega_{j'_1-j_1}} & D^{(l_2)}_{\omega_{j_1+h-j'_2}} & D^{(l_2)}_{\omega_{j'_2-j_1-h}}\\
(Y) & D^{(l_1)}_{\omega_{j'_3-j_2}} & D^{(l_1)}_{\omega_{j_2-j'_3}} & D^{(l_2)}_{\omega_{j'_4-j_2-h}} & D^{(l_2)}_{\omega_{j_2+h-j'_4}}
\end{array}
\tag{S2.12}
$$

In order to satisfy this, in every partition there must be at least one term that contains both an element of X and of Y. A similar reasoning as in the proof of S2.1 indicates we will only have to consider partitions where $2 \leqslant m_i \leqslant 4$ for $i = 1, \dots, M$. In case of stationarity we only have to consider those with $m_i = 2$ for all $i = 1, \dots, M$. In both cases at least one restriction in terms of the summation must occur in order for the partition to be decomposable. In particular, it can be verified that the partition of highest order is of the form

$$
\begin{aligned}
&\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\frac{2\pi}{bT}\Big)^4\sum_{j'_1,j'_2,j'_3,j'_4=1}^{T}\prod_{i=1}^{4}K\Big(\frac{\omega_{j'_i}}{b}\Big)\text{cum}(D^{(l_2)}_{\omega_{j'_4-j_2-h}}D^{(l_2)}_{\omega_{j_2+h-j'_4}})\text{cum}(D^{(l_2)}_{\omega_{j_1+h-j'_2}}D^{(l_2)}_{\omega_{j'_2-j_1-h}})\\
&\qquad\times\text{cum}(D^{(l_1)}_{\omega_{j_1-j'_1}},D^{(l_1)}_{\omega_{j'_3-j_2}})\text{cum}(D^{(l_1)}_{\omega_{j'_1-j_1}},D^{(l_1)}_{\omega_{j_2-j'_3}})\\
&=\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\frac{2\pi}{bT}\Big)^4\sum_{j'_1,j'_2,j'_3,j'_4=1}^{T}\prod_{i=1}^{4}K\Big(\frac{\omega_{j'_i}}{b}\Big)\Big[\tilde{\mathcal{F}}^{(l_2,l_2)}_{0;\omega_{-j_2+h-j'_4}}+O\Big(\frac{1}{T}\Big)\Big]\Big[\tilde{\mathcal{F}}^{(l_2,l_2)}_{0;\omega_{j_1-j'_2+h}}O\Big(\frac{1}{T}\Big)\Big]\\
&\qquad\times\Big[\tilde{\mathcal{F}}^{(l_1,l_1)}_{j_1-j_2-j'_1+j'_3;\omega_{j_1-j'_1}}+O\Big(\frac{1}{T}\Big)\Big]\Big[\tilde{\mathcal{F}}^{(l_1,l_1)}_{j_2-j_1-j'_3+j'_1;\omega_{j'_1-j_1}}+O\Big(\frac{1}{T}\Big)\Big]\\
&\leqslant C\frac{1}{bT}\sup_{\omega}|G^{2(l_2,l_2)}_{\omega}|\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big[\tilde{\mathcal{F}}^{(l_1,l_1)}_{j_1-j_2;\omega_{j_1-j'_1}}+O\Big(\frac{1}{T}\Big)\Big]\Big[\tilde{\mathcal{F}}^{(l_1,l_1)}_{j_2-j_1;\omega_{j'_1-j_1}}+O\Big(\frac{1}{T}\Big)\Big]=O\Big(\frac{1}{bT}\Big),
\end{aligned}
$$

since $\|\tilde{\mathcal{F}}_{j_1-j_2-j'_1+j'_3;\omega_{j_1-j'_1}}\|_2 \leqslant C|j_1-j_2-j'_1+j'_3|^{-2}$ and $\|K(x/b)\|_\infty = O(1)$, where the bandwidth leads to only bT nonzero terms in the summation over j_1. The same bound can be shown to hold under stationarity. As before, the error is uniform with respect to ω which follows again from Corollary A.1 of the main paper. □

Lemma S2.4. *Let* $(g_j^{(l_1,l_2)} : j \in \mathbb{Z})$ *be a bounded sequence in* $\mathbb{C}$ *for all* $l_1, l_2 \in \mathbb{N}$ *such that* $\inf_{l_1,l_2} g_j^{(l_1,l_2)} > 0$. *Then,*

$$\Big|\frac{1}{\sqrt{T}}\sum_{j=1}^{T} g_j^{(l_1,l_2)}\mathbb{E}\big[D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)}\big]\big(\mathbb{E}\big[\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)}\hat{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)}\big] - \mathcal{F}_{\omega_j}^{(l_1,l_1)}\mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)}\big)\Big| = \begin{cases} O\Big(b^2 + \dfrac{1}{bT}\Big) & \textit{under Assumption 4.1.} \\ O\Big(\sqrt{T}b^2 + \dfrac{1}{b\sqrt{T}}\Big) & \textit{under Assumption 4.3.} \end{cases}$$

Proof. First note that

$$\Big|\frac{1}{\sqrt{T}}\sum_{j=1}^{T} g_j^{(l_1,l_2)}\mathbb{E}\big[D_{\omega_j}^{(l_1)}D_{-\omega_{j+h}}^{(l_2)}\big]\Big| = \begin{cases} O\Big(\dfrac{1}{\sqrt{T}}\Big) & \text{under Assumption 4.1.} \\ O(\sqrt{T}) & \text{under Assumption 4.3.} \end{cases}$$

Observe next that

$$\begin{aligned}
&\mathbb{E}\big[\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)}\hat{\mathcal{F}}_{\omega_{j+h}}^{(l_2,l_2)}\big] - \mathcal{F}_{\omega_j}^{(l_1,l_1)}\mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)} \\
&= \Big(\frac{2\pi}{bT}\Big)^2 \sum_{j_1,j_2=1}^{T} K\Big(\frac{\omega_{j_1}}{b}\Big)K\Big(\frac{\omega_{j_2}}{b}\Big)\mathbb{E}\big[D_{\omega_{j-j_1}}^{(l_1)}D_{\omega_{j_1-j}}^{(l_1)}D_{\omega_{j+h-j_2}}^{(l_2)}D_{\omega_{j_2-j-h}}^{(l_2)}\big] - \mathcal{F}_{\omega_j}^{(l_1,l_1)}\mathcal{F}_{\omega_{j+h}}^{(l_2,l_2)} \\
&= \Big(\frac{2\pi}{bT}\Big)^2 \sum_{j_1,j_2=1}^{T} K\Big(\frac{\omega_{j_1}}{b}\Big)K\Big(\frac{\omega_{j_2}}{b}\Big)\Big(\mathrm{cum}(D_{\omega_{j-j_1}}^{(l_1)}, D_{\omega_{j+h-j_2}}^{(l_2)})\mathrm{cum}(D_{\omega_{j_1-j}}^{(l_1)}, D_{\omega_{j_2-j-h}}^{(l_2)}) \\
&\quad + \mathrm{cum}(D_{\omega_{j-j_1}}^{(l_1)}, D_{\omega_{j_2-j-h}}^{(l_2)})\mathrm{cum}(D_{\omega_{j_1-j}}^{(l_1)}, D_{\omega_{j+h-j_2}}^{(l_2)})\Big) + O\Big(b^2 + \frac{1}{bT}\Big).
\end{aligned}$$

Here, it was used that $\mathbb{E}[|\hat{\mathcal{F}}_{\omega_j}^{(l_1,l_1)} - \mathcal{F}_{\omega_j}^{(l_1,l_1)}|] \leqslant \mathbb{E}[\|\hat{\mathcal{F}}_{\omega_j} - \mathcal{F}_{\omega_j}\|_2]\|\psi_{l_1}\|_2\|\psi_{l_1}\|_2 = O(b^2 + 1/bT)$ under H_0. The same bound holds under the alternative, where $\mathcal{F}_\omega$ is replaced with the integrated spectrum G_ω. Under the alternative, write

$$\begin{aligned}
&\Big(\frac{2\pi}{bT}\Big)^2 \sum_{j_1,j_2=1}^{T} K\Big(\frac{\omega_{j_1}}{b}\Big)K\Big(\frac{\omega_{j_2}}{b}\Big)\big(\tilde{\mathcal{F}}_{j_2-j_1-h;\omega_{j-j_1}}^{(l_1,l_2)}\tilde{\mathcal{F}}_{j_1+h-j_2;\omega_{j_1-j}}^{(l_1,l_2)} + \tilde{\mathcal{F}}_{h-j_1-j_2;\omega_{j-j_1}}^{(l_1,l_2)}\tilde{\mathcal{F}}_{j_1+j_2-h;\omega_{j_1-j}}^{(l_1,l_2)} \\
&\quad + O\Big(\frac{1}{T}\Big) + O\Big(b^2 + \frac{1}{bT}\Big) = O\Big(b^2 + \frac{1}{bT}\Big)
\end{aligned}$$

where Corollary A.1 of the main paper was applied and where we used that the bandwidth leads to only bT nonzero terms in the summation. Under H_0, a similar argument shows that the term is of order $O(b^2 + 1/bT)$. The result now follows. □

S3 Convergence of finite-dimensional distributions

Theorem S3.1. *Let Lemma A.1 be satisfied for some finite* $k \geqslant 3$. *Then, for all* $l_i, l_i{}' \in \mathbb{N}$ *and* $h_i \in \mathbb{Z}$ *with* $i = 1, \ldots, k$,

$$\frac{1}{T^{k/2}}\mathrm{cum}\Big(w_{h_1}^{(T)}(\psi_{l_1 l_1{}'}), \ldots, w_{h_k}^{(T)}(\psi_{l_k l_k{}'})\Big) = o(1) \qquad (T \to \infty), \tag{S3.1}$$

where $w_h^{(T)}(\psi_{ll'}) = \langle w_h^{(T)}, \psi_{ll'}\rangle$ *and* $(\psi_{ll'}\colon l,l' \in \mathbb{N})$ *an orthonormal basis of* $L^2([0,1]^2,\mathbb{C})$

Proof. The proof is given in three parts, the first of which provides the outset, the second gives the arguments for the stationary case, while the third deals with the locally stationary situation.

(1) Preliminaries. Fix $\tau_1,\tau_2 \in [0,1]$ and $h = 1,\dots,T-1$. It will be shown that the finite-dimensional distributions of $(w_h^{(T)}(\tau_1,\tau_2)\colon T \in \mathbb{N})$ converge to a Gaussian distribution by proving that the higher order cumulants of the terms $\sqrt{T}w_h^{(T)}(\psi_{ll'}) = \sqrt{T}\langle w_h^{(T)}, \psi_{ll'}\rangle$ vanish asymptotically. To formulate this, consider an array of the form

$$\begin{matrix} (1,1) & (1,2) \\ \vdots & \vdots \\ (k,1) & (k,2) \end{matrix} \tag{S3.2}$$

and let the value $s = ii'$ correspond to entry (i,i'). For a partition $P = \{P_1,\dots,P_Q\}$, the elements of a set P_q will be denoted by $s_{q1},\dots,s_{qm_q}$ where $|P_q| = m_q$ is the corresponding number of elements in P_q. Associate with entry s the frequency index $j_s = j_{ii'} = (-1)^{i'-1}(j_i + h_i^{i'-1})$, Fourier frequency $\lambda_{j_s} = \frac{2\pi j_s}{T}$ and the basis function index $v_s = v_{ii'} = l_i^{2-i'} l_i'^{\,i'-1}$ for $i = 1,\dots,k$ and $i' = 1,2$. An application of the product theorem for cumulants yields

$$\begin{aligned} &\operatorname{cum}\Big(\sum_{j_1=1}^{T} D_{\omega_{j_1}}^{(l_1)} D_{-\omega_{j_1+h_1}}^{(l_1')}, \dots, \sum_{j_k=1}^{T} D_{\omega_{j_k}}^{(l_k)} D_{-\omega_{j_k+h_k}}^{((l_k')}\Big) \\ &\quad = \sum_{j_1,\dots,j_k}\sum_{i.p.} \operatorname{cum}(D_{\lambda_{j_s}}^{(v_s)}\colon s \in P_1)\cdots \operatorname{cum}(D_{\lambda_{j_s}}^{(v_s)}\colon s \in P_Q), \end{aligned}$$

where the summation extends over all indecomposable partitions $P = \{P_1,\dots,P_Q\}$ of (S3.2). Because X_t has zero-mean, the number of elements within each set must satisfy $m_q \geqslant 2$. To ease notation, write $D_{\omega_{j_k}}^{(l)} = \langle D_{\omega_{j_k}}^{(T)}, \psi_l\rangle$ and

$$(\mathcal{F}_{t/T;\lambda_{j_s}}^{(v_s)}\colon s \in P_q) = \langle f_{t/T;\lambda_{j_{q1}},\dots,\lambda_{j_{qm_q-1}}}, \otimes_{i'=1}^{m_q}\psi_{v_{s_{qi'}}}\rangle,$$

noting that the latter quantity is well-defined. An application of Lemma A.1 of the main paper yields

$$\begin{aligned} &\sum_{i.p.}\prod_{q=1}^{Q}\operatorname{cum}\big(D_{\lambda_{j_s}}^{(v_s)}\colon s \in P_q\big) \qquad \text{(S3.3)} \\ &\quad = \sum_{i.p.}\prod_{q=1}^{Q}\Bigg[\frac{(2\pi)^{m_q/2-1}}{T^{m_q/2}}\Bigg(\sum_{t=0}^{T-1}\mathcal{F}_{t/T;\lambda_{j_s}}^{(v_s)} e^{-\mathrm{i}\sum_s t\lambda_{j_s}}\colon s \in P_q\Bigg) + O\Bigg(\frac{1}{T^{m_q/2}}\Bigg)\Bigg], \end{aligned}$$

where under the null $\mathcal{F}_{t/T;\lambda_{j_s}}^{(v_s)} = \mathcal{F}_{\lambda_{j_s}}^{(v_s)}$. In the following, the proof is separated into the cases where the true process is stationary and where it is locally stationary.

(2) Proof under stationarity. Recall that $\sup_{\boldsymbol{\omega}} |||\mathcal{F}_{\omega_{j_1},\dots,\omega_{j_{k'-1}}}|||2 < \infty$ for all $k' \leqslant k$, and thus, by the Cauchy–Schwarz inequality, $\sup_{\boldsymbol{\omega}} |\mathcal{F}_{\lambda_{j_s}}^{(v_s)}| < \infty$ for $s \in P_q$ and $q = 1,\dots,Q$. Therefore,

$$\sum_{i.p.}\prod_{q=1}^{Q}\operatorname{cum}(D_{\lambda_{j_s}}^{(v_s)}\colon s \in P_q)$$

$$\leqslant \sum_{i.p.} \prod_{q=1}^{Q} \Big[\frac{(2\pi)^{m_q/2-1} K_q}{T^{m_q/2}} \Delta^{(T)}\Big(\sum_{s \in P_q} \lambda_{j_s}\big) + O\Big(\frac{1}{T^{m_q/2}}\Big)\Big]$$

for some constants $K_1, \ldots, K_Q$ independent of T. Due to the functions $\Delta^{(T)}$, there are Q constraints if $Q < k$ or if $Q = k$ and there exists h_{i_1} and h_{i_2} such that $h_{i_1} \neq h_{i_2}$ for $i_1, i_2 \in \{1, \ldots, k\}$. On the other hand, if the size of the partition is equal to k and $h_{i_1} = h_{i_2}$ for all $i_1, i_2 \in \{1, \ldots, k\}$, there are $Q - 1$ constraints. This implies that

$$\begin{aligned} &\frac{1}{T^{n/2}} \mathrm{cum}\Big(\sum_{j_1=1}^{T} D^{(l_1)}_{\omega_{j_1}} D^{(l'_1)}_{-\omega_{j_1+h_1}}, \ldots, \sum_{j_n=1}^{T} D^{(l_k)}_{\omega_{j_k}} D^{(l'_k)}_{-\omega_{j_k+h_k}} \Big) \\ &\quad = O(T^{-n/2} T^{n-(Q-1)} T^{-2n/2} T^{Q}) \\ &\quad = O(T^{-n/2+1}). \end{aligned}$$

The cumulants of order $k \geqslant 3$ will therefore tend to 0 as $T \to \infty$.

(3) Proof under local stationarity. Write (S3.3) in terms of the Fourier coefficients as

$$\begin{aligned} &\frac{1}{T^{n/2}} \sum_{j_1,\ldots,j_k=1}^{T} \sum_{i.p.} \prod_{q=1}^{Q} \mathrm{cum}\big(D^{(v_s)}_{\lambda_{k_s}} \colon s \in P_q\big) \\ &\qquad = \frac{1}{T^{n/2}} \sum_{j_1,\ldots,j_k=1}^{T} \sum_{i.p.} \prod_{q=1}^{Q} \Big[\frac{(2\pi)^{m_q/2-1}}{T^{m_q/2-1}} \big(\tilde{\mathcal{F}}^{(v_s)}_{\sum_s j_s; \lambda_{j_s}} \colon s \in P_q\big) + O\Big(\frac{1}{T^{m_q/2}}\Big)\Big]. \end{aligned}$$

Note that, by Corollary A.1 and the Cauchy–Schwarz inequality,

$$\sum_{j=1}^{T} \big|\tilde{\mathcal{F}}^{(v_s)}_{\sum_s j_s; \lambda_{j_s}}\big| \leqslant \sup_{\omega} \sum_{j \in \mathbb{Z}} \|\tilde{\mathcal{F}}_{j;\omega}\|_2 \prod_{i=1}^{m_q} \|\psi_{v_{qi}}\|_2 < \infty, \qquad s \in P_q,$$

for all $q = 1, \ldots, Q$. If $Q < k$ or if $Q = k$ and there are h_{i_1} and h_{i_2} such that $h_{i_1} \neq h_{i_2}$ for $i_1, i_2 \in \{1, \ldots, k\}$ within the same set, then there is dependence on Q of the k sums $j_1, \ldots, j_n$. On the other hand, if the size of the partition is equal to k and $h_{i_1} = h_{i_2}$ for all $i_1, i_2 = 1, \ldots, k$, then there are $Q-1$ constraints on $j_1, \ldots, j_n$. Thus, similar to the stationary case, it follows that the order is

$$O(T^{-k/2} T^{k-Q+1} T^{-2k/2+Q}) = O(T^{-k/2+1}),$$

hence giving the result. □

S4 Weak convergence

In this section, the asymptotic properties of the statistic $\hat{\gamma}^{(T)}$ are derived. Theorems 4.1 and 4.5 imply that for appropriate choices of the bandwidths the analysis of distributional properties may be restricted to $\gamma^{(T)}$ under both hypotheses. Because the denominator of $\gamma^{(T)}$ is deterministic, the analysis may be further simplified by focusing on the properties of the numerator

$$w_h^{(T)}(\tau_1,\tau_2) = \frac{1}{T}\sum_{j=1}^{T} D_{\omega_j}^{(T)}(\tau_1) D_{-\omega_{j+h}}^{(T)}(\tau_2).$$

To demonstrate weak convergence, it is useful to apply a result from Cremers & Kadelka (1986) as it considerably simplifies the verification of the usual tightness condition often invoked in weak convergence proofs. In particular, the following lemma indicates that weak convergence of the functional process will almost directly follow from the weak convergence of the finite dimensional distributions once it is weakly tight in a certain sense.

Lemma S4.1. *Let $(T, \mathcal{B}, \mu)$ be a measure space, let $(E, |\cdot|)$ be a Banach space, and let $X = (X_n \colon n \in \mathbb{N})$ be a sequence of random elements in $L_E^p(T,\mu)$ such that*

(i) the finite–dimensional distributions of X converge weakly to those of a random element X_0 in $L_E^p(T,\mu)$;

(ii) $\limsup_{n\to\infty} \mathbb{E}[\|X_n\|_p^p] \leqslant \mathbb{E}[\|X_0\|_p^p]$.

Then, X converges weakly to X_0 in $L_E^p(T,\mu)$.

To apply the lemma in the present context, consider the sequence $(\hat{E}_r^{(T)}(\tau_1,\tau_2) \colon T \in \mathbb{N})$ of random elements in $L^2([0,1]^2,\mathbb{C})$, for $\tau,\tau' \in [0,1]$ and $h = 1,\ldots,T-1$ defined through

$$\hat{E}_h^{(T)}(\tau_1,\tau_2) = \sqrt{T}\Big(w_h^{(T)}(\tau_1,\tau_2) - \mathbb{E}\big[w_h^{(T)}(\tau_1,\tau_2)\big]\Big).$$

Let $(\psi_l \colon l \in \mathbb{N})$ be an orthonormal basis of $H_{\mathbb{C}}$ and denote the elementary tensor product by $\psi_{ll'} = \psi_l \otimes \psi_l'$. Then $(\psi_{ll'} \colon l,l' \in \mathbb{N})$ forms an orthonormal basis of $L^2([0,1]^2,\mathbb{C})$ and a basis expansion yields

$$\hat{E}_h^{(T)} = \sum_{l,l'=1}^{\infty} \langle \hat{E}_h^{(T)}, \psi_{ll'}\rangle \psi_{ll'}.$$

It can therefore be seen that the finite–dimensional distributions of the basis coefficients provide a complete characterization of the distributional properties of $\hat{E}_r^{(T)}$: Weak convergence of $(\langle \hat{E}_h^{(T)}, \psi_{ll'}\rangle \colon l,l' \in \mathbb{N})$ in the sequence space $\ell_{\mathbb{C}}^2$ will imply weak convergence of the process $(\hat{E}_r^{(T)}(\tau_1,\tau_2) \colon T \in \mathbb{N})$. Identifying the functional $\hat{E}_h^{(T)}$ with its dual $(\hat{E}_h^{(T)})^* \in L^2([0,1]^2,\mathbb{C})^*$, leads to the pairing

$$\hat{E}_h^{(T)}(\phi) = \langle \hat{E}_h^{(T)}, \phi\rangle$$

for all $\phi \in L^2([0,1]^2,\mathbb{C})^*$. The second condition of Lemma S4.1 will hence be satisfied if

$$\mathbb{E}\big[\|\hat{E}_h^{(T)}\|_2^2\big] = \sum_{l,l'=1}^{\infty} \mathbb{E}\big[\big|\hat{E}_h^{(T)}(\psi_{ll'})\big|^2\big] \to \sum_{l,l'=1}^{\infty} \mathbb{E}\big[\big|E_h(\psi_{ll'})\big|^2\big] = \mathbb{E}\big[\|E_h\|_2^2\big] \qquad (T\to\infty), \tag{S4.1}$$

with E_h denoting the limiting process. The following theorem shows that the finite–dimensional distributions converge weakly to a Gaussian process and is a restatement of the results of Section S3.

Theorem S4.1. *Under Assumption 4.1, for all $l_i, l_i' \in \mathbb{N}$, $h_i = 1, \ldots, T-1$, $i = 1, \ldots, k$ and $k \geqslant 3$,*

$$\operatorname{cum}\Big(\hat{E}_{h_1}^{(T)}(\psi_{l_1 l_1'}), \ldots, \hat{E}_{h_k}^{(T)}(\psi_{l_k l_k'})\Big) = o(1) \qquad (T \to \infty). \tag{S4.2}$$

The higher order cumulant result of Theorem S4.1 establishes, for all $q \in \mathbb{N}$, the joint convergence of $\hat{E}_h^{(T)}(\psi_{l_1 l_1'}), \ldots, \hat{E}_h^{(T)}(\psi_{l_q l_q'})$ and condition (i) of Lemma S4.1 is therefore satisfied. Weak convergence of the functional process can now be determined, distinguishing between the real and imaginary parts.

Theorem S4.2 (Weak convergence under the null). *Let $(X_t \colon t \in \mathbb{Z})$ be a stochastic process taking values in $H_{\mathbb{R}}$ satisfying Assumption 4.1 with $k = 8$ and $\ell = 2$. Then,*

$$\big(\Re\hat{E}_{h_i}^{(T)}, \Im\hat{E}_{h_i}^{(T)} \colon i = 1, \ldots, k\big) \xrightarrow{D} (\mathcal{R}_{h_i}, \mathcal{I}_{h_i} \colon i = 1, \ldots, k), \tag{S4.3}$$

where $\mathcal{R}_{h_1}, \mathcal{I}_{h_2}$, $h_1, h_2 \in \{1, \ldots, T-1\}$, are jointly Gaussian elements in $L^2([0,1]^2, \mathbb{C})$ with means $\mathbb{E}[\mathcal{R}_{h_1}(\psi_{ll'})] = \mathbb{E}[\mathcal{I}_{h_2}(\psi_{ll'})] = 0$ and covariances

$$\begin{aligned}
\operatorname{Cov}\big(&\mathcal{R}_{h_1}(\psi_{l_1 l_1'}), \mathcal{R}_{h_2}(\psi_{l_2 l_2'})\big) && \text{(S4.4)}\\
&= \operatorname{Cov}\big(\mathcal{I}_{h_1}(\psi_{l_1 l_1'}), \mathcal{I}_{h_2}(\psi_{l_2 l_2'})\big)\\
&= \frac{1}{4\pi}\int_0^{2\pi} \big\langle \mathcal{F}_{\omega}(\psi_{l_2}), \psi_{l_1}\big\rangle \big\langle \mathcal{F}_{-\omega-\omega_h}(\psi_{l_2'}), \psi_{l_1}\big\rangle \, d\omega && \text{(S4.5)}\\
&\quad + \frac{1}{4\pi}\int_0^{2\pi} \big\langle \mathcal{F}_{\omega}(\psi_{l_2'}), \psi_{l_1}\big\rangle \big\langle \mathcal{F}_{-\omega-\omega_h}(\psi_{l_2}), \psi_{l_1'}\big\rangle \, d\omega\\
&\quad + \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \big\langle \mathcal{F}_{\omega,-\omega-\omega_h,-\omega'}(\psi_{l_2 l_2'}), \psi_{l_1 l_1'}\big\rangle \, d\omega d\omega'
\end{aligned}$$

for all $h_1 = h_2$ and l_1, l_1', l_2, l_2', and 0 otherwise. In addition,

$$\operatorname{Cov}(\mathcal{R}_{h_1}(\psi_{l_1 l_1'}), \mathcal{I}_{h_2}(\psi_{l_2 l_2'})) = 0$$

uniformly in h_1, h_2 and l_1, l_1', l_2, l_2'.

Proof. The covariance structure follows from Theorem 4.2 and the convergence of the finite–dimensional distributions from Theorem S4.1. It then remains to verify that the condition (ii) of Lemma S4.1 is satisfied. This follows from Theorem 4.2, since

$$\mathbb{E}\big[\|\hat{E}_h^{(T)}\|_2^2\big] = \int_{[0,1]^2} \operatorname{Var}\big(\hat{E}_h^{(T)}(\tau, \tau')\big) d\tau d\tau' = T\|\operatorname{Var}(w_h^{(T)})\|_2^2 = 2\|\operatorname{Var}(\mathcal{R}_h)\|_2^2.$$

This completes the proof. □

Under the alternative, a similar result is obtained.

Theorem S4.3 (Weak convergence under the alternative). *Let $(X_t \colon t \in \mathbb{Z})$ be a stochastic process taking values in $H_{\mathbb{R}}$ satisfying Assumption 4.3 with $k = 8$ and $\ell = 2$. Then,*

$$\big(\Re\hat{E}^{(T)}_{h_i}, \Im\hat{E}^{(T)}_{h_i} \colon i = 1, \dots, k\big) \xrightarrow{d} (\mathcal{R}_{h_i}, \mathcal{J}_{h_i} \colon i = 1, \dots, k), \tag{S4.6}$$

where $\mathcal{R}_{h_1}, \mathcal{J}_{h_2}$, $h_1, h_2 \in \{1, \dots, T-1\}$, are jointly Gaussian elements in $L^2([0,1]^2, \mathbb{C})$ with means $\mathbb{E}[\mathcal{R}_{h_1}(\psi_{ll'})] = \mathbb{E}[\mathcal{J}_{h_2}(\psi_{ll'})] = 0$ and covariance structure

1. $\mathrm{Cov}(\mathcal{R}_{h_1}(\psi_{l_1 l_1'}), \mathcal{R}_{h_2}(\psi_{l_2 l_2'})) =$
$$\frac{1}{4}\big[\Upsilon_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) + \acute{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) + \grave{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) + \bar{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'})\big]$$
2. $\mathrm{Cov}(\mathcal{R}_{h_1}(\psi_{l_1 l_1'}), \mathcal{J}_{h_2}(\psi_{l_2 l_2'})) =$
$$\frac{1}{4\mathrm{i}}\big[\Upsilon_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) - \acute{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) + \grave{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) - \bar{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'})\big]$$
3. $\mathrm{Cov}(\mathcal{J}_{h_1}(\psi_{l_1 l_1'}), \mathcal{J}_{h_2}(\psi_{l_2 l_2'})) =$
$$\frac{1}{4}\big[\Upsilon_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) - \acute{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) - \grave{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'}) + \bar{\Upsilon}_{h_1,h_2}(\psi_{l_1 l_1'\, l_2 l_2'})\big]$$

for all h_1, h_2 and l_1, l_1', l_2, l_2', and where Υ_{h_1,h_2}, $\acute{\Upsilon}_{h_1,h_2}$, $\grave{\Upsilon}_{h_1,h_2}$ and $\bar{\Upsilon}_{h_1,h_2}$ are given in (S5.5)–(S5.7).

Proof. The covariance structure follows along the lines of Theorem 4.6 and the convergence of the finite–dimensional distributions from Theorem S4.1. Condition (ii) of Lemma S4.1 is satisfied by Theorem 4.6 since

$$\mathbb{E}\big[\|\hat{E}^{(T)}_h\|_2^2\big] = \int_{[0,1]^2} \mathrm{Var}\big(\hat{E}^{(T)}_h(\tau,\tau')\big)d\tau d\tau' = T\|\mathrm{Var}(w^{(T)}_h)\|_2^2 = \|\mathrm{Var}(\mathcal{R}_h)\|_2^2 + \|\mathrm{Var}(\mathcal{J}_h)\|_2^2,$$

which completes the proof. □

S5 Proofs under the alternative hypothesis of local stationarity

Completion of the proof of Theorem 4.6. To find the expressions for the covariance structure of $\sqrt{T}\gamma^{(T)}_h$ and its complex conjugate, use equation (B.2) and Lemma A.1 of the main paper to write

$$\begin{aligned}
&\mathrm{Cov}\big(D^{(l_1)}_{\omega_{j_1}} D^{(l_2)}_{-\omega_{j_1+h_1}}, D^{(l_3)}_{\omega_{j_2}} D^{(l_4)}_{-\omega_{j_2+h_2}}\big)\\
&\quad= \frac{2\pi}{T}\tilde{\mathfrak{F}}^{(l_1\, l_2, l_3\, l_4)}_{(-h_1+h_2;\omega_{j_1},-\omega_{j_1+h_1},-\omega_{j_2})} + O\Big(\frac{1}{T^2}\Big)\\
&\qquad+ \Big[\tilde{\mathfrak{F}}^{(l_1,l_3)}_{(j_1-j_2;\omega_{j_1})} + O\Big(\frac{1}{T}\Big)\Big]\Big[\tilde{\mathfrak{F}}^{(l_2,l_4)}_{(-j_1-h_1+j_2+h_2;-\omega_{j_1+h_1})} + O\Big(\frac{1}{T}\Big)\Big]\\
&\qquad+ \Big[\tilde{\mathfrak{F}}^{(l_1,l_4)}_{(j_1+j_2+h_2;\omega_{j_1})} + O\Big(\frac{1}{T}\Big)\Big]\Big[\tilde{\mathfrak{F}}^{(l_2,l_3)}_{(-j_1-h_1-j_2,-\omega_{j_1+h_1})} + O\Big(\frac{1}{T}\Big)\Big].
\end{aligned}$$

Thus,

$$\mathrm{Cov}\big(\sqrt{T}\gamma^{(T)}_{h_1}(l_1,l_2), \sqrt{T}\gamma^{(T)}_{h_2}(l_3,l_4)\big)$$

$$= \frac{1}{T}\sum_{j_1,j_2=1}^{T}\Bigg\{\frac{2\pi}{T}\frac{\tilde{\mathcal{F}}^{(l_1\,l_2,l_3\,l_4)}_{(-h_1+h_2;\omega_{j_1},-\omega_{j_1+h_1},-\omega_{j_2})}}{(G^{(l_1,l_1)}_{\omega_{j_1}}G^{(l_2,l_2)}_{-\omega_{j_1+h_1}}G^{(l_3,l_3)}_{-\omega_{j_2}}G^{(l_4,l_4)}_{\omega_{j_2+h_2}})^{1/2}} + \frac{\tilde{\mathcal{F}}^{(l_1,l_3)}_{(j_1-j_2;\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_4)}_{(-j_1-h_1+j_2+h_2;-\omega_{j_1+h_1})}}{(G^{(l_1,l_1)}_{\omega_{j_1}}G^{(l_3,l_3)}_{-\omega_{j_2}}G^{(l_2,l_2)}_{-\omega_{j_1+h_1}}G^{(l_4,l_4)}_{\omega_{j_2+h_2}})^{1/2}}$$
$$+ \frac{\tilde{\mathcal{F}}^{(l_1,l_4)}_{(j_1+j_2+h_2;\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_3)}_{(-j_1-h_1-j_2,-\omega_{j_1+h_1})}}{(G^{(l_1,l_1)}_{\omega_{j_1}}G^{(l_4,l_4)}_{\omega_{j_2+h_2}}G^{(l_2,l_2)}_{-\omega_{j_1+h_1}}G^{(l_3,l_3)}_{-\omega_{j_2}})^{1/2}}$$
$$+ O\Bigg(\frac{1}{T}\Bigg[\frac{\tilde{\mathcal{F}}^{(l_1,l_3)}_{(j_1-j_2;\omega_{j_1})}}{(G^{(l_1,l_1)}_{\omega_{j_1}}G^{(l_3,l_3)}_{-\omega_{j_2}})^{1/2}} + \frac{\tilde{\mathcal{F}}^{(l_2,l_4)}_{(-j_1-h_1+j_2+h_2;-\omega_{j_1+h_1})}}{(G^{(l_2,l_2)}_{-\omega_{j_1+h_1}}G^{(l_4,l_4)}_{\omega_{j_2+h_2}})^{1/2}}$$
$$+ \frac{\tilde{\mathcal{F}}^{(l_1,l_4)}_{(j_1+j_2+h_2;\omega_{j_1})}}{(G^{(l_1,l_1)}_{\omega_{j_1}}G^{(l_4,l_4)}_{\omega_{j_2+h_2}})^{1/2}} + \frac{\tilde{\mathcal{F}}^{(l_2,l_3)}_{(-j_1-h_1-j_2,-\omega_{j_1+h_1})}}{(G^{(l_2,l_2)}_{-\omega_{j_1+h_1}}G^{(l_3,l_3)}_{-\omega_{j_2}})^{1/2}}\Bigg] + \frac{1}{T^2}\Bigg)\Bigg\}.$$

By Corollary A.1 (ii), this equals

$$\begin{aligned}\Sigma^{(T)}_{h_1,h_2}(\boldsymbol{l}_4) &= T\,\mathrm{Cov}\big(\gamma^{(T)}_{h_1}(l_1,l_2),\gamma^{(T)}_{h_2}(l_3,l_4)\big) \\ &= \frac{1}{T}\sum_{j_1,j_2=1}^{T}\mathcal{G}^{(l_1,l_2,l_3,l_4)}_{j_1,j_2}\Big(\tilde{\mathcal{F}}^{(l_1,l_3)}_{(j_1-j_2;\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_4)}_{(-j_1-h_1+j_2+h_2;-\omega_{j_1+h_1})} \\ &\quad + \tilde{\mathcal{F}}^{(l_1,l_4)}_{(j_1+j_2+h_2;\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_3)}_{(-j_1-h_1-j_2,-\omega_{j_1+h_1})} + \frac{2\pi}{T}\tilde{\mathcal{F}}^{(l_1\,l_2,l_3\,l_4)}_{(-h_1+h_2;\omega_{j_1},-\omega_{j_1+h_1},-\omega_{j_2})}\Big) \\ &\quad + O\Big(\frac{1}{T}\Big),\end{aligned} \tag{S5.1}$$

where $\mathcal{G}^{(l_1,l_2,l_3,l_4)}_{j_1,j_2} = (G^{(l_1,l_1)}_{\omega_{j_1}}G^{(l_2,l_2)}_{\omega_{j_1+h_1}}G^{(l_3,l_3)}_{\omega_{j_2}}G^{(l_4,l_4)}_{\omega_{j_2+h_2}})^{-1/2}$. Similarly,

$$\begin{aligned}\acute{\Sigma}^{(T)}_{h_1,h_2}(\boldsymbol{l}_4) &= T\,\mathrm{Cov}\big(\gamma^{(T)}_{h_1}(l_1,l_2),\overline{\gamma^{(T)}_{h_2}(l_3,l_4)}\big) \\ &= \frac{1}{T}\sum_{j_1,j_2=1}^{T}\mathcal{G}^{(l_1,l_2,l_3,l_4)}_{j_1,j_2}\Big(\tilde{\mathcal{F}}^{(l_1,l_3)}_{(j_1+j_2;\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_4)}_{(-j_1-h_1-j_2-h_2;-\omega_{j_1+h_1})} \\ &\quad + \tilde{\mathcal{F}}^{(l_1,l_4)}_{(j_1-j_2-h_2;\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_3)}_{(-j_1-h_1+j_2;-\omega_{j_1+h_1})} + \frac{2\pi}{T}\tilde{\mathcal{F}}^{(l_1\,l_2,l_3\,l_4)}_{(-h_1-h_2;\omega_{j_1},-\omega_{j_1+h_1},\omega_{j_2})}\Big) \\ &\quad + O\Big(\frac{1}{T}\Big),\end{aligned} \tag{S5.2}$$

$$\begin{aligned}\bar{\Sigma}^{(T)}_{h_1,h_2}(\boldsymbol{l}_4) &= T\,\mathrm{Cov}\big(\overline{\gamma^{(T)}_{h_1}(l_1,l_2)},\overline{\gamma^{(T)}_{h_2}(l_3,l_4)}\big) \\ &= \frac{1}{T}\sum_{j_1,j_2=1}^{T}\mathcal{G}^{(l_1,l_2,l_3,l_4)}_{j_1,j_2}\Big(\tilde{\mathcal{F}}^{(l_1,l_3)}_{(-j_1+j_2;-\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_4)}_{(j_1+h_1-j_2-h_2;\omega_{j_1+h_1})} \\ &\quad + \tilde{\mathcal{F}}^{(l_1,l_4)}_{(-j_1-j_2-h_2;-\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_3)}_{(j_1+h_1+j_2;\omega_{j_1+h_1})} + \frac{2\pi}{T}\tilde{\mathcal{F}}^{(l_1\,l_2,l_3\,l_4)}_{(h_1-h_2;-\omega_{j_1},\omega_{j_1+h_1},\omega_{j_2})}\Big) \\ &\quad + O\Big(\frac{1}{T}\Big),\end{aligned} \tag{S5.3}$$

and

$$\begin{aligned}\grave{\Sigma}^{(T)}_{h_1,h_2}(\boldsymbol{l}_4) &= T\,\mathrm{Cov}\big(\overline{\gamma^{(T)}_{h_1}(l_1,l_2)},\gamma^{(T)}_{h_2}(l_3,l_4)\big) \\ &= \frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\mathcal{G}^{(l_1,l_2,l_3,l_4)}_{j_1,j_2}\tilde{\mathcal{F}}^{(l_1,l_3)}_{(-j_1-j_2;-\omega_{j_1})}\tilde{\mathcal{F}}^{(l_2,l_4)}_{(j_1+h_1+j_2+h_2;\omega_{j_1+h_1})}\end{aligned} \tag{S5.4}$$

$$+ \tilde{\mathfrak{F}}^{(l_1,l_4)}_{(-j_1+j_2+h_2;-\omega_{j_1})}\tilde{\mathfrak{F}}^{(l_2,l_3)}_{(j_1+h_1-j_2;\omega_{j_1+h_1})} + \frac{2\pi}{T}\tilde{\mathfrak{F}}^{(l_1\,l_2,l_3\,l_4)}_{(h_1+h_2;-\omega_{j_1},\omega_{j_1+h_1},-\omega_{j_2})}\Big)$$
$$+ O\Big(\frac{1}{T}\Big).$$

This completes the proof. □

Similarly, we find for the covariance structure of Theorem S4.3:

$$\Upsilon_{h_1,h_2}(\psi_{l_1 l'_1\, l_2 l'_2}) = \lim_{T\to\infty}\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\big\langle\tilde{\mathfrak{F}}_{j_1-j_2;\omega_{j_1}}(\psi_{l_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{-j_1-h_1+j_2+h_2;-\omega_{j_1+h_1}}(\psi_{l'_2}),\psi_{l'_1}\big\rangle$$
$$+ \big\langle\tilde{\mathfrak{F}}_{j_1+j_2+h_2;\omega_{j_1}}(\psi_{l'_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{-j_1-h_1-j_2,-\omega_{j_1+h_1}}(\psi_{l_2}),\psi_{l'_1}\big\rangle$$
$$+ \frac{2\pi}{T}\big\langle\tilde{\mathfrak{F}}_{(-h_1+h_2;\omega_{j_1},-\omega_{j_1+h_1},-\omega_{j_2})}(\psi_{l_2\,l'_2}),\psi_{l_1,l'_1}\big\rangle\Big), \tag{S5.5}$$

$$\acute{\Upsilon}_{h_1,h_2}(\psi_{l_1 l'_1\, l_2 l'_2}) = \lim_{T\to\infty}\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\big\langle\tilde{\mathfrak{F}}_{j_1+j_2;\omega_{j_1}}(\psi_{l_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{-j_1-h_1-j_2-h_2;-\omega_{j_1+h_1}}(\psi_{l'_2}),\psi_{l'_1}\big\rangle$$
$$+ \big\langle\tilde{\mathfrak{F}}_{j_1-j_2-h_2;\omega_{j_1}}(\psi_{l'_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{-j_1-h_1+j_2;-\omega_{j_1+h_1}}(\psi_{l_2}),\psi_{l'_1}\big\rangle$$
$$+ \frac{2\pi}{T}\big\langle\tilde{\mathfrak{F}}_{(-h_1-h_2;\omega_{j_1},-\omega_{j_1+h_1},\omega_{j_2})}(\psi_{l_2\,l'_2}),\psi_{l_1\,l'_1}\big\rangle\Big), \tag{S5.6}$$

$$\bar{\Upsilon}_{h_1,h_2}(\psi_{l_1 l'_1\, l_2 l'_2}) = \lim_{T\to\infty}\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\big\langle\tilde{\mathfrak{F}}_{-j_1+j_2;-\omega_{j_1}}(\psi_{l_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{j_1+h_1-j_2-h_2;\omega_{j_1+h_1}}(\psi_{l'_2}),\psi_{l'_1}\big\rangle$$
$$+ \big\langle\tilde{\mathfrak{F}}_{-j_1-j_2-h_2;-\omega_{j_1}}(\psi_{l'_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{j_1+h_1+j_2;\omega_{j_1+h_1}}(\psi_{l_2}),\psi_{l'_1}\big\rangle$$
$$+ \frac{2\pi}{T}\big\langle\tilde{\mathfrak{F}}_{(h_1-h_2;-\omega_{j_1},\omega_{j_1+h_1},\omega_{j_2})}(\psi_{l_2\,l'_2}),\psi_{l_1\,l'_1}\big\rangle\Big) \tag{S5.7}$$

and

$$\grave{\Upsilon}_{h_1,h_2}(\psi_{l_1 l'_1\, l_2 l'_2}) = \lim_{T\to\infty}\frac{1}{T}\sum_{j_1,j_2=1}^{T}\Big(\big\langle\tilde{\mathfrak{F}}_{-j_1-j_2;-\omega_{j_1}}(\psi_{l_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{j_1+h_1+j_2+h_2;\omega_{j_1+h_1}}(\psi_{l'_2}),\psi_{l'_1}\big\rangle$$
$$+ \big\langle\tilde{\mathfrak{F}}_{-j_1+j_2+h_2;-\omega_{j_1}}(\psi_{l'_2}),\psi_{l_1}\big\rangle\big\langle\tilde{\mathfrak{F}}_{j_1+h_1-j_2;\omega_{j_1+h_1}}(\psi_{l_2}),\psi_{l'_1}\big\rangle$$
$$+ \frac{2\pi}{T}\big\langle\tilde{\mathfrak{F}}_{(h_1+h_2;-\omega_{j_1},\omega_{j_1+h_1},-\omega_{j_2})}(\psi_{l_2\,l'_2}),\psi_{l_1\,l'_1}\big\rangle\Big) \tag{S5.8}$$

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