HW #15 (Chapter 4): Answers

1.1 We have:
   (i) \( P(X = 4, Y = 0) = f_{x,y}(4, 0) = 0.10. \)
   (ii) \( P(X = 5) = f_{x,y}(5, 0) + f_{x,y}(5, 1) + f_{x,y}(5, 2) \)
       \( = 0.05 + 0.03 + 0.02 = 0.10. \)
   (iii) \( P(Y = 1) = f_{x,y}(0, 1) + f_{x,y}(1, 1) + f_{x,y}(2, 1) + f_{x,y}(3, 1) \)
       \( + f_{x,y}(4, 1) + f_{x,y}(5, 1) \)
       \( = 0.015 + 0.030 + 0.075 + 0.090 + 0.060 + 0.030 = 0.30. \)
   (iv) \( P(X \leq 3, Y \geq 1) = f_{x,y}(0, 1) + f_{x,y}(0, 2) + f_{x,y}(1, 1) + f_{x,y}(1, 2) \)
       \( + f_{x,y}(2, 1) + f_{x,y}(2, 2) + f_{x,y}(3, 1) + f_{x,y}(3, 2) \)
       \( = 0.015 + 0.010 + 0.020 + 0.075 + 0.050 + 0.090 + 0.060 = 0.35. \)

1.2 We have:

\[
\begin{array}{c|ccc}
    & 0 & 1 & 2 \\
\hline
0 & 0.3 & 0.2 & 0.075 \\
1 & 0 & 0.2 & 0.15 \\
2 & 0 & 0 & 0.075 \\
\end{array}
\]

\( P(X = x, Y = y) = P(Y = y|X = x)P(X = x) \), and observe that:

\( P(X = 0, Y = 1) = P(X = 0, Y = 2) = P(X = 1, Y = 2) = 0. \) For the
remaining values of \( x \) and \( y \), we obtain:

\( P(X = 0, Y = 0) = P(Y = 0|X = 0)P(X = 0) = 1 \times 0.3 = 0.3, \)
\( P(X = 1, Y = 0) = P(Y = 0|X = 1)P(X = 1) = 0.5 \times 0.4 = 0.2, \)
\( P(X = 1, Y = 1) = P(Y = 1|X = 1)P(X = 1) = 0.5 \times 0.4 = 0.2, \)
\( P(X = 2, Y = 0) = P(Y = 0|X = 2)P(X = 2) = (0.5)^2 \times 0.3 = 0.075, \)
\( P(X = 2, Y = 1) = P(Y = 1|X = 2)P(X = 2) = 0.5 \times 0.3 = 0.15, \)
\( P(X = 2, Y = 2) = P(Y = 2|X = 2)P(X = 2) = (0.5)^2 \times 0.3 = 0.075. \)

1.3 Since \( P(X < Y) + P(Y < X) = 1 \), and the p.d.f. is symmetric with respect to \( x \) and \( y \), one would expect that \( P(X < Y) = 0.5. \) Indeed

\[
P(X < Y) = \int_0^1 \int_0^y f_{x,y}(x, y) \, dx \, dy = \int_0^1 \int_0^y (x + y) \, dx \, dy
\]

\[
= \int_0^1 \int_0^y \left( \frac{x^2}{2} + y \right) \, dy \, dx = \frac{3}{2} \int_0^1 y^2 \, dy = \frac{3}{2} \cdot \frac{y^3}{3} \bigg|_0^1 = \frac{3}{2} = 0.5. \]

\[ \blacksquare \]
1.5 (i) \( P(X \leq Y \leq c) = \int_{\substack{x \leq y \leq c}} e^{-x^2-y^2} \, dx \, dy = \int_{0}^{c} \int_{0}^{y} e^{-x^2-y^2} \, dx \, dy \)

\[
= \int_{0}^{c} e^{-y^2} \left( \int_{0}^{y} e^{-x^2} \, dx \right) \, dy = \int_{0}^{c} e^{-y^2} \left( e^{-x^2} \right|_{0}^{y} \right) \, dy = \int_{0}^{c} e^{-y^2} \left( 1 - e^{-y^2} \right) \, dy = \int_{0}^{c} e^{-y^2} \, dy - \int_{0}^{c} e^{-x^2} \, dx = -e^{-y^2} \bigg|_{0}^{c} + \frac{1}{2} e^{-2y^2} \bigg|_{0}^{c} = \frac{1}{2} - \frac{1}{c^2} + \frac{1}{2e^c}.
\]

(ii) For \( c = \log 2 \), we have: \( P(X \leq Y \leq \log 2) = \frac{1}{6} = 0.125. \]

1.7 From

\[
\int_{0}^{c} \int_{0}^{y} \frac{2}{c} \, dx \, dy = \frac{2}{c^2} \int_{0}^{c} \int_{0}^{y} \, dx \, dy = \frac{2}{c^2} \int_{0}^{c} y \, dy = \frac{2}{c^2} \left[ \frac{y^2}{2} \right]_{0}^{c} = 2 \frac{c^2}{2} = 1,
\]

we have that \( c \) can be any positive constant. \( \blacksquare \)

1.10 We have:

For \( 0 < x \leq 1 \), \( \int_{1-x}^{2-x} cx \, dx = cx \), and for \( 1 < x \leq 2 \),

\[
\int_{0}^{2-x} cx \, dy = cx(2 - x).
\]

Since \( \int_{0}^{1} cx \, dx = \frac{c}{2} \) and

\[
\int_{1}^{2} cx(2 - x) \, dx = \frac{2c}{3}, \text{ we have } \frac{c}{2} + \frac{2c}{3} = 1 \text{ and } c = \frac{6}{7}. \]

\( \blacksquare \)
2.3 We have:
\[ f_X(-4) = f_X(-2) = f_X(2) = f_X(4) = 0.25; \]
and
\[ f_Y(-2) = f_Y(-1) = f_Y(1) = f_Y(2) = 0.25. \]

2.9 We have:
(i)
\[
\begin{align*}
  f_X(x) &= \int_{0}^{\infty} \frac{xe^{-x}}{2} \, dx, \quad x > 0; \\
  f_Y(y) &= \int_{0}^{\infty} ye^{-y} \, dx = ye^{-y}, \quad y > 0.
\end{align*}
\]
(ii)
\[
\begin{align*}
  f_{X|Y}(x|y) &= \frac{ye^{-y}}{x^2e^{-x}}, \quad 0 < y \leq x < \infty; \\
  f_{Y|X}(y|x) &= \frac{2y}{x^2e^{-x}}, \quad 0 < y \leq x < \infty;
\end{align*}
\]
(iii)
\[
\begin{align*}
  P(X > 2 \log 2 | Y = \log 2) &= \int_{2\log 2}^{\infty} xe^{-x} \, dx = -2e^{-x}_{\log 4} \\
  &= 2e^{-\log 4} = \frac{2}{4} = 0.5.
\end{align*}
\]

2.11 We have:
(i) \[ f_Y(y) = \int_{0}^{\infty} ye^{-yx} \, dx = \frac{1}{y} \int_{0}^{\infty} ye^{-v} \, dv = \frac{1}{y}, \quad 0 < y < 2, \]
either by integration or by observing that \( ye^{-x}, x > 0 \) is the p.d.f. of the Negative Exponential p.d.f. with parameter \( y \).
(ii) \[ f_{X|Y}(x|y) = \frac{1}{y} ye^{-yx} = ye^{-yx}, \quad x > 0, \quad 0 < y < 2, \]
which is the Negative Exponential p.d.f. with parameter \( y \). Also, \( f_{X|Y}(x|\frac{1}{y}) = \frac{1}{x}e^{-x/2}, x > 0 \).
(iii) \[ E(X|Y = y) = \int_{0}^{\infty} x \times ye^{-yx} \, dx = \frac{1}{y} \]
either by integration or by observing that \( ye^{-x}, x > 0 \) is the p.d.f. of the Negative Exponential p.d.f. with parameter \( y \).

2.15 We have:
(i) \[ f_X(x) = \int_{0}^{y} xy^2 \, dy = \frac{1}{2}x^2, \quad 0 < x < c_1, \]
f_Y(y) = \int_{0}^{y} xy^2 \, dx = \frac{1}{2}x^2, \quad 0 < y < c_2.
(ii) \[ f_{X|Y}(x|y) = \frac{2x}{c_1^2}, \quad 0 < x < c_1, \quad 0 < y < c_2. \]
(iii) We have: \[ EX = \int_{0}^{c_1} x \times \frac{2x}{c_1^2} \, dx = \frac{2}{3} \times \frac{c_1^3}{3} = \frac{2c_1^3}{9}; \]
\[ E(Y) = \frac{2c_1^3}{9} \]
\[ E(X|Y = y) = \int_{0}^{c_1} x \times \frac{2x}{c_1^2} \, dx = \frac{2}{3} \times \frac{c_1^3}{3} = \frac{2c_1^3}{9}, \quad 0 < y < c_2. \]
(iv) \[ E[E(X|Y)] = \int_{0}^{y} \frac{2c_1^3}{9} \times \frac{2x}{c_1^2} \, dy = \frac{2}{3} \times \frac{c_1^3}{3} = \frac{2c_1^3}{9} = EX. \]
3.7 We have: 

(i) \[ \begin{align*} 
EX &= (-4 - 2 + 2 + 4) \times 0.25 = 0, \\
EY &= (-2 - 1 + 1 + 2) \times 0.25 = 0, \\
Var(X) &= EX^2 = (16 + 4 + 4 + 16) \times 0.25 = 10, \\
Var(Y) &= EY^2 = (4 + 1 + 1 + 4) \times 0.25 = 2.5. 
\end{align*} \]

(ii) \[ \begin{align*} 
E(XY) &= [(-4) \times 1 + (-2) \times (-2) + 2 \times 2 + 4 \times (-1)] \times 0.25 \\
&= 0 \times 0.25 = 0, \text{ so that} \\
\text{Cov}(X, Y) &= 0 \text{ and hence } \rho(X, Y) = 0. 
\end{align*} \]

(iii) \( \rho(X, Y) = 0 \) implies that the r.v.'s \( X \) and \( Y \) are uncorrelated. This is borne out by the graph, which shows that the four points are irregularly spread in the plane and have no resemblance to any degree of linear arrangement.

3.10 We have:

\[ \begin{align*} 
EX &= \frac{1}{4} (-2 - 1 + 1 + 2) = 0, \Var(X) &= EX^2 = \frac{1}{4} (4 + 1 + 1 + 4) = \frac{10}{3}, \\
EY &= \frac{1}{4} (4 + 1 + 1 + 4) = \frac{10}{4} = \frac{5}{2}, \Var(Y) &= EY^2 = \frac{1}{4} (16 + 1 + 1 + 16) = \frac{34}{4}, \\
\text{so that } \text{Cov}(X, Y) &= 0 \text{ and } \rho(X, Y) = 0. 
\end{align*} \]

The fact that \( Y \) is completely determined by \( X \) through the relation \( Y = X^2 \) and yet \( \rho(X, Y) = 0 \) seems peculiar from the first viewpoint. However, it is entirely correct and appropriate, since \( \rho(X, Y) = 0 \) states that there is no linear relationship between \( X \) and \( Y \) of whatever degree. Indeed, their relationship is quadratic.
3.14 For convenience, set $EX = \mu$ and $\text{Var}(X) = \sigma^2$. Then

$EY = a\mu + b$, $\text{Var}(Y) = a^2\sigma^2$, and $E(XY) = E(aX^2 + bX)$

$= aEX^2 + b\mu = a(\sigma^2 + \mu^2) + b\mu = a\sigma^2 + a\mu^2 + b\mu$.

Then

$\text{Cov}(X, Y) = a\sigma^2 + a\mu^2 + b\mu - \mu(a\mu + b)$

$= a\sigma^2 + a\mu^2 + b\mu - a\mu^2 - b\mu = a\sigma^2$, and

$\rho(X, Y) = \frac{a\sigma^2}{\sigma|a|\sigma} = \frac{a}{|a|}$. Thus, $|\rho(X, Y)| = 1$,

and $\rho(X, Y) = 1$ if and only if $a > 0$ (so $a = |a|$), and $\rho(X, Y) = -1$ if and only if $a < 0$ (so that $a = -|a|$).

3.15 Indeed,

(i) $P(UV < 0) = P((X + Y)(X - Y) < 0) = P(X^2 - Y^2 < 0)$

$= P(X^2 < Y^2)$

$= P(|X| < |Y|)$.

(ii) $E(UV) = E(X^2 - Y^2) = EX^2 - EY^2 = 0$.

(iii) $\text{Cov}(U, V) = E(UV) - (EU)(EV)$

$= E(X^2 - Y^2) - (EX + EY)(EX - EY)$

$= EX^2 - EY^2 - [(EX)^2 - (EY)^2] = [EX^2 - (EX)^2] - [EY^2 - (EY)^2]$]

$= \text{Var}(X) - \text{Var}(Y) = 0$,

so that $U$ and $V$ are uncorrelated.