3.27 Indeed, with \( \alpha > 1 \),
\[
\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy = -\int_0^\infty y^{\alpha-1}de^{-y} = -\int_0^\infty y^{\alpha-2}e^{-y}dy
\]
\[
+ (\alpha-1) \int_0^\infty y^{\alpha-2}e^{-y}dy
\]
\[
= 0 + (\alpha-1) \int_0^\infty y^{(\alpha-1)-1}e^{-y}dy = (\alpha-1)\Gamma(\alpha-1).
\]

3.31 Here \( f(x) = \lambda e^{-\lambda x}, x > 0 \), so that:

(i) \( F(x) = \int_0^x \lambda e^{-\lambda t}dt = -e^{-\lambda t} \bigg|_0^x = 1 - e^{-\lambda x} \), and \( 1 - F(x) = e^{-\lambda x} \).
Therefore
\[
r(x) = \frac{f(x)}{1 - F(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - 1} = \lambda, \text{ constant for all } x > 0.
\]

(ii) \( P(X > s + t|X > t) = \frac{P(X > s + t, X > t)}{P(X > t)} = \frac{P(X > s + t)}{P(X > t)} = \frac{1 - F(s + t)}{1 - F(t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda t} \text{ independent of } t > 0.
\]

The independence from \( t \) says that the underlying distribution is "memoryless." This is not a particularly desirable attribute if this distribution is to serve as a lifetime distribution.

3.34 (i) Observe that:
\[
\int_0^\infty \alpha x^{\beta-1}e^{-ax}dx = \int_0^\infty e^{-ax}(\alpha x^{\beta-1})dx = -\int_0^\infty de^{-ax}
\]
\[
= -e^{-ax} \bigg|_0^\infty = -\frac{1}{\alpha}dx = -e^{-ax}. 
\]

(ii) For \( \beta = 1 \) and any \( \alpha > 0 \).

(iii) For \( n = 1, 2, \ldots \),
\[
EX^n = \alpha \beta \int_0^\infty x^n \times x^{\beta-1}e^{-ax}dx = \alpha \beta \int_0^\infty x^{n+\beta-1}e^{-ax}dx.
\]

Set \( ax = t \), so that \( x = \frac{t}{\alpha \beta}, dx = \frac{1}{\alpha \beta}dt \) and \( 0 < t < \infty \). Then:
\[
EX^n = \alpha \beta \int_0^\infty \frac{t^{\beta+n}}{\alpha \beta} \times \frac{1}{\alpha \beta}e^{-t}dt
\]
\[
= \frac{\Gamma(\beta+n+1)}{\alpha^n \beta^n} \int_0^\infty \frac{1}{\Gamma(\beta+1)}t^{\beta+n+1-1}e^{-t}dt = \frac{\Gamma(\beta+n+1)}{\alpha^n \beta^n}.
\]

because \( \frac{1}{\Gamma(\beta+1)}t^{\beta+n+1-1}e^{-t}(t > 0) \), is the p.d.f. of the Gamma distribution with parameters \( \frac{\beta+n}{2} + 1 \) and 1. Thus, \( EX^n = \Gamma(\beta+1)/\alpha^n \beta^n \).

For \( n = 1 \) and \( n = 2 \), we get \( EX = \Gamma(\frac{1}{2}+1)/\alpha^{1/2} \beta \) and \( EX^2 = \Gamma(\frac{3}{2}+1)/\alpha^{1/2} \beta^2 \), respectively, so that
\[
Var(X) = \frac{\Gamma(\frac{3}{2}+1)}{\alpha^{1/2} \beta^2} - \left[ \frac{\Gamma(\frac{1}{2}+1)}{\alpha^{1/2}} \right]^2 = \frac{\Gamma(\frac{3}{2}+1) - \left[ \Gamma(\frac{1}{2}+1) \right]^2}{\alpha^{1/2} \beta^2}.
\]
From Exercise 3.34(i),

\[ F(x) = \int_0^x a\beta t^{\beta-1} e^{-at} dt = -e^{-ax} \bigg|_0^x = 1 - e^{-ax}, \]

so that, for \( x > 0 \),

\[ 1 - F(x) = e^{-ax} \text{ and } r(x) = \frac{f(x)}{1 - F(x)} = \frac{a\beta x^{\beta-1} e^{-ax}}{e^{-ax}} = a\beta x^{\beta-1}. \]

\[ P(X > s + t | X > t) = \frac{P(X > s + t, X > t)}{P(X > t)} = \frac{P(X > s + t)}{P(X > t)} \]

\[ = \frac{1 - F(s + t)}{1 - F(t)} = \frac{e^{-a(s+t)}}{e^{-at}} = e^{-a(s+t-t)}. \]

(iii) Here, both the failure rate and the conditional survival probability do depend on the variables involved. This is a desirable characteristic for lifetime distributions.
3.37 By part (iv) of Exercise 3.35:
(i) \( P(-1 < Z < 1) = 2\Phi(1) - 1 = 2 \times 0.841345 - 1 = 0.68269. \)
(ii) \( P(-2 < Z < 2) = 2\Phi(2) - 1 = 2 \times 0.977250 - 1 = 0.9545. \)
(iii) \( P(-3 < Z < 3) = 2\Phi(3) - 1 = 2 \times 0.998650 - 1 = 0.9973. \)

3.38 Clearly,
(i) \[ P(X < c) = P(X \leq c) = 2 - 9[1 - P(X \leq c)] = -7 + 9P(X \leq c), \]
    so that\[ P(X \leq c) = \frac{7}{8} \text{ or } P\left(\frac{X - \mu}{\sigma} \leq \frac{c - \mu}{\sigma}\right) = \frac{7}{8} = 0.875, \]
and hence (from the Normal Tables) \( \frac{c - \mu}{\sigma} = 1.15 \) or \( c = \mu + 1.15\sigma. \)
(ii) Here \( c = 5 + 1.15 \times 2 = 7.30. \)

3.39 We have:
\[ P(|X - \mu| < k\sigma) \geq 1 - \frac{\sigma^2}{(k\sigma)^2} = 1 - \frac{1}{k^2}. \]
Therefore, for \( k = 1, 2, 3, \) the respective bounds are: 0 (meaningless); \( \frac{1}{4} = 0.75, \) which is about 78.6\% of the exact probability (0.9545) in the normal case; \( \frac{1}{9} = 0.8889, \) which is about 89.1\% of the exact probability (0.9973) in the normal case.

3.41 Let us term as a success the fact that a tested light bulb turns out to be defective. Assuming that the testings are independent, we have that the number of defective light bulbs is a r.v. \( Y \sim B(25, p), \) where \( p = P(X < 1.800). \) We have
\[ p = P\left(\frac{X - \mu}{\sigma} \leq \frac{1.800 - 2.000}{200}\right) = P(Z \leq -1) = 1 - P(Z \leq 1) \]
\[ = 1 - \Phi(1) = 1 - 0.841345 = 0.158655, \]
and therefore
\[ P(\text{at most 15 light bulbs are defective}) = P(Y \leq 15) = \sum_{y=0}^{15} \binom{25}{y}(0.158655)^y(0.841345)^{25-y} \]
\[ \geq \sum_{y=0}^{15} \binom{25}{y}(0.150)^y(0.841)^{25-y}. \]

3.47 Observe that \( M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \) with \( \mu = \alpha \) and \( \sigma^2 = 2\beta, \) which is the m.g.f. of the \( N(\alpha, \sigma^2) \) distribution. Thus, \( X \sim N(\alpha, 2\beta). \)

3.52 In the first place, \( EX = \int_0^\infty x f_X(x) dx = \frac{x^2}{2} \int_0^1 = \frac{1}{3} \) and \( EX^2 = \int_0^\infty x^2 f_X(x) dx = \frac{x^3}{3} \int_0^1 = \frac{1}{3}, \) so that:
(i) \( E(3X^2 - 7X + 2) = 3EX^2 - 7EX + 2 = 3 \times \frac{1}{3} - 7 \times \frac{1}{3} + 2 = -0.5. \)
(ii) \( E(2e^X) = 2Ee^X = 2 \int_0^\infty e^x dx = 2e^\frac{1}{2} = 2(e - 1) \approx 3.44. \)
4.1 (i) $EX = \int_0^1 \frac{3}{2} x^2 \, dx = \frac{3}{2} \cdot \frac{1}{3} = 0.75$, $m = \int_0^{0.5} 2x^2 \, dx = 0.5$, so that $x_{2.5} = (0.5)^{1/3} \approx 0.794$, and mean < median.
(ii) $\int_0^{0.125} 3x^2 \, dx = x_{0.125}^3 = 0.125 = \frac{1}{8}$, so that $x_{0.125} = \frac{1}{2} = 0.5$.

4.3 (i) From $P(X \leq x_p) = \int_0^{x_p} e^{-\lambda x} \, dx = -e^{-\lambda x}|_0^{x_p} = 1 - e^{-\lambda x_p} = p$, we get $e^{-\lambda x_p} = 1 - p$, or $x_p = \frac{-\ln(1-p)}{\lambda}$.
(ii) For $\lambda = 1/10$ and $p = 0.25$, we have: $x_{0.25} = -10 \log(0.75) \approx 2.877$.

4.6 Here

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>10</th>
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</table>

(ii) $EX = \frac{1}{36}(2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4 + 6 \times 5 + 7 \times 6 + 8 \times 5 + 9 \times 4 + 10 \times 3 + 11 \times 2 + 12 \times 1) = \frac{2525}{36} \approx 7$.

(iii) The median and the mode are the same here and equal to 7.

4.7 (i) Mode = 1 and the maximum is 1/2.
(ii) Mode = 1 and the maximum is $1 - \alpha$; $\alpha = \frac{1}{2}$.
(iii) Mode = 0 and the maximum is 2/3.

4.8 Here $(n + 1)p = \frac{125}{2} = 25.25$ is not an integer. Then there is a unique mode obtained for $x = 25$. In other words, $f(25)$ is the unique maximum probability among the probabilities $f(x) = P(X = x), x = 0, 1, \ldots, 100$, and one would bet on the value $X = 25$. 

\[\boxrule=5pt\boxdepth=0\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
x & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
f(x) & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} \\
\hline
\end{array}\boxrule=5pt\boxdepth=0\]