1.5 The values of $X$ are: 1, 2, 3, 4, 5, 6, each taken with probability $1/6$. Thus:

(i) $M_X(t) = \sum_{x=1}^{6} e^{tx} \times \frac{1}{6} = \frac{1}{6}(e^{t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}), \quad t \in \mathbb{R}$.

(ii) $EX = \frac{d}{dt}M_X(t) \bigg|_{t=0} = \frac{1}{6}(e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}) \bigg|_{t=0} = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3.5$;

$EX^2 = \frac{d^2}{dt^2}M_X(t) \bigg|_{t=0} = \frac{1}{6}(e^{2t} + 4e^{3t} + 9e^{4t} + 16e^{5t} + 25e^{6t} + 36e^{7t}) \bigg|_{t=0} = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$, so that

$Var(X) = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{105}{36}$, and s.d. of $X = \sqrt{\frac{105}{6}} \approx 1.708$.

1.9 We have:

(i) $EX = \sum_{x=0}^{\infty} x \times c(\frac{1}{3})^x = c \sum_{x=0}^{\infty} x(\frac{1}{3})^x = \frac{c}{3} \sum_{x=0}^{\infty} (\frac{1}{3})^{x-1}$. At this point, consider $\sum_{x=0}^{\infty} x(\frac{1}{3})^x = \sum_{x=1}^{\infty} \frac{t^{x-1}}{x!} = \frac{d}{dt} \sum_{x=0}^{\infty} t^x = \frac{d}{dt} \frac{1}{1-t}$ (for $|t| < 1$), i.e., Setting $t = \frac{1}{1-x}$, we get: $EX = \frac{c}{3} \times \frac{1}{\left(\frac{1}{1-x}\right)^2} = \frac{c}{1-x}$.

In Exercise 2.10 in Chapter 2, it was seen that $c = \frac{1}{3}$. Therefore $EX = \frac{1}{9}$.

(ii) $M_X(t) = \sum_{x=0}^{\infty} e^{tx} \times c(\frac{1}{3})^x = c \sum_{x=0}^{\infty} (\frac{1}{3})^x = c \times \frac{1}{1-\frac{1}{3}}$, provide $\frac{1}{3} < 1$ or $t < \log 3$. Thus, $M_X(t) = \frac{3}{2}e^t$, $t < \log 3$.

(iii) $EX = \frac{d}{dt} \left(\frac{3}{2}e^t\right) \bigg|_{t=0} = \frac{3}{2}e^t \bigg|_{t=0} = \frac{3}{2}$.

1.15 In the first place, this is a p.d.f. for any $c > 0$, since

\[
\int_{-\infty}^{\infty} \frac{|x|}{c^2} dx = \int_{-\infty}^{0} \frac{|x|}{c^2} dx + \int_{0}^{\infty} \frac{|x|}{c^2} dx = \frac{1}{c^2} \int_{-\infty}^{0} x dx + \frac{1}{c^2} \int_{0}^{\infty} x dx
\]

\[= -\frac{1}{c^2} \times \frac{x^2}{2} \bigg|_{-c}^{0} + \frac{1}{c^2} \times \frac{x^2}{2} \bigg|_{0}^{\infty} = \frac{1}{c^2} \times \frac{c^2}{2} + \frac{1}{c^2} \times \frac{c^2}{2} = 1. \]

Next,

\[EX^n = \frac{1}{c^2} \int_{-\infty}^{\infty} x^n |x| dx = \frac{1}{c^2} \int_{-\infty}^{0} x^n |x| dx + \frac{1}{c^2} \int_{0}^{\infty} x^n |x| dx \]

\[= -\frac{1}{c^2} \int_{-\infty}^{0} x^{n+1} dx + \frac{1}{c^2} \int_{0}^{\infty} x^{n+1} dx \]

\[= -\frac{1}{c^2} \times \frac{x^{n+2}}{n+2} \bigg|_{-\infty}^{-c} + \frac{1}{c^2} \times \frac{x^{n+2}}{n+2} \bigg|_{0}^{\infty} = \frac{1}{c^2} \times (-c)^{n+2} + \frac{1}{c^2} \times \frac{c^{n+2}}{n+2} = \frac{1}{(n+2)c^2} [(-c)^{n+2} + (c)^{n+2}]. \]

Therefore:

\[EX^n = \begin{cases} \frac{1}{(n+2)c^2} \times 2c^{n+2} = \frac{2c^n}{n+2}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd}. \end{cases} \]

In particular, $EX = 0$ and $EX^2 = \frac{c^2}{1}$, so that $Var(X) = \frac{c^2}{3}$. ■
2.1 (i) \( EX = -1 \times \frac{1}{15} + 0 \times \frac{6}{15} + 1 \times \frac{8}{15} = 0, \) \( EX^2 = \frac{1}{5}, \) so that \( \text{Var}(X) = \frac{1}{5} \) and s.d. of \( X = \frac{\sqrt{2}}{5}. \) So, \( \mu = 0, \sigma = 1/3. \)

(ii) \( P(|X - \mu| \geq k\sigma) = P(|X| \geq \frac{k}{\sigma}) = \frac{3}{15} = \frac{1}{5} \approx 0.111, \) both for \( k = 2 \) and \( k = 3. \)

(iii) \( P(|X - \mu| \geq k\sigma) \leq \frac{x^2}{2\sigma^2} = \frac{1}{k^2}. \) For \( k = 2, \frac{1}{4} = 0.25 \) (more than twice as large the exact probability), and for \( k = 3, \frac{1}{9} = \frac{1}{3} \) (exactly the same as the exact probability).

2.3 From Exercise 2.11(i) in Chapter 2, \( c = 3/4. \) Next,

(i) \( EX = c \int_{-1}^{1} x(1-x^2)dx = c \left( \frac{x^2}{2} \right|_{-1}^{1} - \frac{x^4}{4} \right) = 0, \)

\[ EX^2 = c \int_{-1}^{1} x^2(1-x^2)dx = c \left( \frac{x^3}{3} \right|_{-1}^{1} - \frac{x^5}{5} \right) = \frac{4c}{15} = \frac{4}{15} \times \frac{3}{4} = \frac{1}{5} = 0.2, \] so that \( \text{Var}(X) = 0.2. \)

(ii) \( P(-0.9 < X < 0.9) = P(|X| < 0.9) = P(|X - EX| < 0.9) \geq 1 - \frac{0.9^2}{0.2^2} \approx 1 - 0.247 = 0.753. \)

In Exercise 2.11(ii) of Chapter 2, it was seen that the exact probability \( P(-0.9 < X < 0.9) = 0.9855. \) Thus, the approximate probability 0.753 is about 76.4% of the exact probability.
3.3 If X is the r.v. denoting the number of tickets to win a prize, then $X \sim B(50, 0.01)$. Therefore:

(i) $P(X = 1) = \binom{50}{1}(0.01)(0.99)^{49} \approx 0.306$.

(ii) $P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{50}{0}(0.01)^0(0.99)^{50} \approx 1 - 0.605 = 0.395$.

3.5 If X is the r.v. denoting the number of successes in 18 tossings, then $X \sim B(18, 1/6)$. Since the number of failures is 18 - X, we have:

(i) $P(X > 18 - X) = P(2X > 18) = P(X > 9) = 1 - P(X \leq 9)$

$= 1 - \sum_{x=0}^{9} \binom{18}{x} \left( \frac{1}{6} \right)^x \left( \frac{5}{6} \right)^{18-x} \approx 0.000114$.

(ii) $P(X = 2(18 - X)) = P(X = 36 - 2X) = P(3X = 36) = P(X = 12)$

$= \binom{18}{12} \left( \frac{1}{6} \right)^{12} \left( \frac{5}{6} \right)^6 \approx 0.000002856$.

(iii) $P(18 - X = 3X) = P(4X = 18) = P(X = 4.5) = 0$.

3.7 Here $EX = np = \frac{100}{4} = 25$ and $Var(X) = npq = 25 \times \frac{3}{4} = 18.75$, so that:

$P(|X - 25| < 10) = P(|X - EX| < 10) \geq 1 - \frac{\sigma^2(X)}{100} = 1 - \frac{18.75}{100}$

$= 1 - 0.1875 = 0.8125$.

3.8 Here $E\left(\frac{X}{n}\right) = \frac{np}{n} = p$ and $Var\left(\frac{X}{n}\right) = \frac{npq}{n^2} = \frac{pq}{n}$. Thus, $P\left(\frac{X}{n} - p \leq 0.05, \sqrt{pq}\right)$

$\geq 1 - \frac{5}{\sqrt{npq}} = 1 - \frac{500}{n}$, and suffices $1 - \frac{500}{n} \geq 0.95$, or $n \geq 8,000$.

3.13 Here X has the Geometric distribution with $p = 0.01$, so that $f(x) = (0.01)(0.99)^{x-1}$, $x = 1, 2, \ldots$. Then:

$P(X \leq 10) = 1 - P(X \geq 11) = 1 - (0.01)(0.99)^{10}[1 + 0.99 + (0.99)^2 + \cdots]$

$= 1 - (0.01)(0.99)^{10} \times \frac{1}{1 - 0.99} = 1 - (0.99)^{10} \approx 0.006$.

3.15 If X is the r.v. denoting the smallest number of tosses until the first head appears, then X has the Geometric distribution with parameter $p$. Then:

(i) The probability that the first head appears by the nth time is $\sum_{x=1}^{n} pq^{x-1}$ and we require the determination of the smallest value of n for which:

$\sum_{x=1}^{n} pq^{x-1} \geq \alpha$, or $1 - \sum_{x=n+1}^{\infty} pq^{x-1} \geq \alpha$, or $1 - q^n \geq \alpha$, or

$n \geq \log(1 - \alpha)/\log q$.

(ii) For $\alpha = 0.95$, and $p = 0.25$ ($q = 0.75$) and $p = 0.50(=q)$, we obtain: $n \geq 10.42$ and $n \geq 4.33$, respectively, so that: $n = 11$, and $n = 5$, respectively.
3.19 Since $P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}, x = 0, 1, \ldots$, we have: $P(X = 0) = e^{-\lambda} = 0.1$.
so that $-\lambda = \log(0.1) = -2.3$ and $\lambda = 2.3$. Then:
$$P(X = 5) = e^{-\lambda} \frac{(2.3)^5}{5!} \approx 0.054.$$ 

2.25 (i) The distribution of $X$ is Hypergeometric with $m = 15$ and $n = 10$, so that:
$$f(x) = \binom{15}{x} \frac{\binom{10}{x-1}}{\binom{25}{x}}, \quad x = 5, \ldots, 15.$$

(ii) $P(X \geq 10) = \sum_{x=10}^{15} \binom{15}{x} \frac{\binom{10}{x-1}}{\binom{25}{x}}$.

(iii) This probability is 0, since there are only 10 specimens from the rock $R_2$.  

3.26 We have:
$$f(x + 1) = \frac{\binom{m}{x+1} \binom{n}{r-x-1}}{\binom{m+n}{r}}$$
$$= \frac{1}{\binom{m+n}{r}} \times \frac{m!}{(x+1)! (m-x-1)!} \times \frac{n!}{(r-x)! (n-r+x+1)!} \times \frac{\binom{m}{x+1} \binom{n}{r-x-1}}{\binom{m+n}{r}}$$
$$= \frac{1}{\binom{m+n}{r}} \times \frac{m-x}{x+1} \times \frac{x!(m-x)}{x!(m-x)} \times \frac{r-x}{n-r+x+1} \times \frac{n!}{(r-x)! (n-r+x)!}$$
$$= \frac{(m-x)(r-x)}{(m-r+x+1)(x+1)} \times \frac{\binom{m}{x+1} \binom{n}{r-x-1}}{\binom{m+n}{r}} = \frac{(m-x)(r-x)}{(m-r+x+1)(x+1)} f(x).$$