HW # 23 (Chapter 6): Answers

5.1 (i) For \( x \geq 1 \), \( F(x) = \int_1^x e^{-t} \, dt = -e^{-t}\Big|_1^x = 1 - \frac{1}{e^x} \).

So, \( F(x) = 0 \) for \( x \leq 1 \), and \( F(x) = 1 - \frac{1}{x} \) for \( x > 1 \).

(ii) In formulas (28) and (29), take \( a = 1, b = \infty \) to obtain.

\[
    f_U(u) = n \times \left( \frac{1}{\sqrt{c}} \right)^{n-1} \times u^{-c} = n c u^{-c+1}, \quad u > 1,
\]

and

\[
    f_V(v) = n \left( 1 - \frac{1}{\sqrt{c}} \right)^{n-1} \times v^{-b} = n c (v^b - 1)^{n-1} v^{-b-1}, \quad v > 1. \quad \blacksquare
\]

5.2 Here \( g_1(y) = n(1 - y)^{n-1}, 0 < y < 1; g_2(y_n) = n y_n^{n+1}, 0 < y_n < 1 \).

Therefore

\[
    EY_1 = \int_0^1 y \times n(1 - y)^{n-1} \, dy = -\int_0^1 y \, dy = -\frac{1}{2} = \frac{1}{n+1}; \quad \text{i.e.,} \quad EY_1 = \frac{1}{n+1}.
\]

Also,

\[
    EY_n = \int_0^1 y \times n y_n^{n+1} \, dy = n \int_0^1 y \, dy = \frac{n}{n+1} \quad \text{i.e.,} \quad EY_n = \frac{n}{n+1}.
\]

Then \( \lim_{n \to \infty} EY_n = 1. \quad \blacksquare \)
Statistics 120 Spring Quarter 2006

HW #24 (Chapter 7): Answers

1.1 For any \( x \in \mathbb{R} \), take \( n_0 > x \). Then \( F_n(x) = 0 \) for all \( n \geq n_0 \), and hence, trivially, \( F_n(x) \to 0 \) for all \( x \in \mathbb{R} \).

2.3 (i) Here \( P(X \leq 80) = \sum_{i=0}^{100} (100)^i (0.6)^i (0.4)^{100-i} \).
(ii) For this part, look at \( X \) as the sum \( \sum_{i=1}^{100} X_i \) of independent r.v.'s \( X_1, \ldots, X_{100} \) distributed as \( B(1, 0.6) \). Then the CLT applies and gives:

\[
P(X \leq 80) = \mathbb{P}(0 \leq X \leq 80) = P\left[ \frac{-150 \times 0.6}{100 \times 0.6} \leq \frac{X - EX}{\text{Var}(X)} \leq \frac{80 - 150 \times 0.6}{100 \times 0.6} \right] \\
= P\left[ -0.6 \leq \frac{X - EX}{\sqrt{\text{Var}(X)}} \leq -1.5 \right] \approx \Phi(-1.67) - \Phi(-15) \\
= \Phi(-1.67) - 1 - \Phi(1.67) = 0.952540 - 0.04746 = 0.905080.
\]

2.5 Setting \( X = \sum_{i=1}^{100} X_i \), we have \( X \sim B(100, p) \) with \( EX = 100p \) and \( \text{Var}(X) = 100pq \) \((q = 1 - p)\). Then:

(i) \( P(X = 50) = \binom{100}{50} p^{50} q^{50} \).
(ii) Clearly,

\[
P(X = 50) = P(49.5 \leq X \leq 50.5) = P\left[ \frac{49.5 - 100p}{\sqrt{100pq}} \leq \frac{X - EX}{\text{s.d.}(X)} \leq \frac{50.5 - 100p}{\sqrt{100pq}} \right] \\
= \Phi\left( \frac{56.5 - 100p}{100\sqrt{pq}} \right) - \Phi\left( \frac{49.5 - 100p}{100\sqrt{pq}} \right).
\]

(iii) For \( p = 0.5 \), the probability in part (ii) becomes:

\[
\Phi(0.1) - \Phi(-0.1) = 2\Phi(0.1) - 1 = 2 \times 0.539828 - 1 = 0.079656.
\]

2.7 As usual, associate the r.v. \( X_i \) with the \( i \)th draw, where \( X_i = 1 \) when an ace appears and \( X_i = 0 \) otherwise. Then the r.v.'s \( X_1, \ldots, X_{100} \) may be assumed to be independent and their common distribution is the \( B(1, p) \) with \( p = \frac{1}{13} \). Furthermore, \( X = \sum_{i=1}^{100} X_i \). Since \( EX_i = \frac{1}{13} \), \( \text{Var}(X_i) = \frac{4}{13} \times \frac{12}{13} = \frac{48}{169} \), we have \( EX = \frac{100}{13} \), \( \text{Var}(X) = \frac{4800}{169} \) and \( \text{s.d}(X) = \frac{60}{13} \). Therefore

\[
P(65 \leq X \leq 90) \geq P\left[ \frac{65 - \frac{100}{13}}{\frac{60}{13}} \leq Z \leq \frac{90 - \frac{100}{13}}{\frac{60}{13}} \right] \\
= P\left( \frac{-31\sqrt{30}}{120} \leq Z \leq \frac{17\sqrt{30}}{60} \right) = \Phi\left( \frac{17\sqrt{30}}{60} \right) - \Phi\left( \frac{31\sqrt{30}}{120} \right) = 1.
2.9 (i) Since $X = \sum_{i=1}^{n} X_i$, where the independent r.v.'s $X_1, \ldots, X_n$ are distributed as $B(1, p)$, so that $E X_i = p$, $\text{Var}(X_i) = pq$, we have that $\frac{X}{n} = \bar{X}$ and hence:

\[
P\left(\left| \frac{X}{n} - p \right| < 0.05\sqrt{pq} \right) = P\left(\left| \bar{X} - p \right| < 0.05\sqrt{pq} \right)
\]

\[
= P(-0.05\sqrt{pq} < \bar{X} - p < 0.05\sqrt{pq})
\]

\[
= P\left[ \frac{-0.05\sqrt{n}}{\sqrt{pq}} \sqrt{n}(\bar{X} - p) < \frac{0.05\sqrt{n}}{\sqrt{pq}} \sqrt{n}(\bar{X} - p) < \frac{0.05\sqrt{n}}{\sqrt{pq}} \sqrt{n}(\bar{X} - p) \right]
\]

\[
= P\left[ -0.05\sqrt{n} < \sqrt{n}(\bar{X} - p) < 0.05\sqrt{n} \right]
\]

\[
\cong \Phi(0.05\sqrt{n}) - \Phi(-0.05\sqrt{n}) = 2\Phi(0.05\sqrt{n}) - 1 = 0.95 \quad \text{or} \quad \Phi(0.05\sqrt{n}) = 0.975,
\]

so that $0.05\sqrt{n} = 1.96$ and $n = 1,536.64$, so that $n = 1,537$.

(ii) Since

\[
P\left(\left| \frac{X}{n} - p \right| < 0.05\sqrt{pq} \right) = P\left(\left| \bar{X} - p \right| < 0.05\sqrt{pq} \right)
\]

\[
\geq 1 - \frac{\text{Var}(\bar{X})}{(0.05\sqrt{pq})^2} = 1 - \frac{pq/n}{0.0025pq} = 1 - \frac{1}{0.0025n}
\]

\[
= 1 - \frac{400}{n}, \quad \text{it suffices for} \quad 1 - \frac{400}{n} \geq 0.95 \quad \text{or} \quad n = 8,000.
\]

(iii) The sample size determined through the CLT is only about 19.2% of that determined by way of the Tchebichev inequality.

2.14 The total lifetime of 50 tubes is the r.v. $X = \sum_{i=1}^{50} X_i$, where $X_i$ has the Negative Exponential distribution with parameter $\lambda = 1/1,500$. Since $EX_i = \frac{1}{\lambda} = 1,500$, $\text{Var}(X_i) = \frac{\lambda^2}{\lambda^2} = 1,500^2$, then $EX = 50 \times 1,500$ and $\text{Var}(X) = 50 \times 1,500^2$.

Therefore

\[
P(X \geq 80,000) = P\left[ \frac{X - EX}{\text{s.d.}(X)} \geq \frac{80,000 - 50 \times 1,500}{50 \times 1,500^2} \right]
\]

\[
\cong P(Z \geq 0.47) = 1 - \Phi(0.47) = 1 - 0.680722 = 0.31928.
\]

2.20 (i) It is assumed implicitly that $n$ is large enough, so that the CLT applies. Then:

\[
P(|\bar{X} - \mu| < k\sigma) = P(-k\sigma < \bar{X} - \mu < k\sigma)
\]

\[
= P\left[ -k\sqrt{n} < \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} < k\sqrt{n} \right] \cong \Phi(k\sqrt{n}) - \Phi(-k\sqrt{n})
\]

\[
= 2\Phi(k\sqrt{n}) - 1 = p \quad \text{or} \quad \Phi(k\sqrt{n}) = \frac{1 + p}{2},
\]

and hence

\[
k\sqrt{n} = \Phi^{-1}\left(\frac{1 + p}{2}\right) \quad \text{or} \quad n = \left[\frac{1}{k}\Phi^{-1}\left(\frac{1 + p}{2}\right)\right]^2.
\]

Thus, $n$ is the smallest integer which is $\geq \left[\frac{1}{k}\Phi^{-1}\left(\frac{1 + p}{2}\right)\right]^2$. 

(ii) Since $E\bar{X}_n = \mu$ and $\text{Var}(\bar{X}_n) = \sigma^2/n$, we have:

$$P(|\bar{X}_n - \mu| < k\sigma) \geq 1 - \frac{k^2/n}{\sigma^2} = 1 - \frac{k^2}{\sigma^2}. \text{ Then it suffices for } 1 - \frac{k^2}{\sigma^2} \geq p \text{ or } n \geq \frac{1}{1 - p}k^2. \text{ Thus, } n \text{ is the smallest integer which is } \geq \frac{1}{1 - p}k^2.$$

(iii) For $p = 0.90$, $\Phi^{-1}(\frac{1-p}{2}) = \Phi^{-1}(0.050) = 1.645$ (by interpolation).

For $p = 0.95$, $\Phi^{-1}(\frac{1-p}{2}) = \Phi^{-1}(0.075) = 1.96$.

For $p = 0.99$, $\Phi^{-1}(\frac{1-p}{2}) = \Phi^{-1}(0.095) = 2.575$ (by interpolation).

Then, for the various values of $k$, the respective values of $n$ are given in the following table for part (i):

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.90</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>11</td>
<td>16</td>
<td>27</td>
</tr>
<tr>
<td>0.25</td>
<td>44</td>
<td>62</td>
<td>167</td>
</tr>
<tr>
<td>0.10</td>
<td>271</td>
<td>385</td>
<td>664</td>
</tr>
</tbody>
</table>

For the Chebyshev inequality, the values of $n$ are given by the entries of the table below:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0.90</th>
<th>0.95</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>40</td>
<td>80</td>
<td>400</td>
</tr>
<tr>
<td>0.25</td>
<td>160</td>
<td>320</td>
<td>1,600</td>
</tr>
<tr>
<td>0.10</td>
<td>1,000</td>
<td>2,000</td>
<td>10,000</td>
</tr>
</tbody>
</table>