Continuous Random Variables

• Continuous random variables can assume the infinitely many values corresponding to points on a line interval.

• Examples:
  – Heights, weights
  – Length of life of a particular product
  – Experimental laboratory error

Properties of Continuous Probability Distributions

• The area under the curve is equal to 1.

• \( P(a \leq x \leq b) = \text{area under the curve} \) between a and b.

• There is no probability attached to any single value of \( x \). That is, \( P(x = a) = 0 \).
The Normal Distribution

- The formula that generates the normal probability distribution is:
  \[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \text{ for } -\infty < x < \infty \]
  
  \[ e = 2.7183 \quad \pi = 3.1416 \]
  
  \[ \mu \text{ and } \sigma \text{ are the population mean and standard deviation.} \]

- The shape and location of the normal curve changes as the mean and standard deviation change.

\[ \text{APPLET} \]

The Standard Normal Distribution

- To find \( P(a < x < b) \), we need to find the area under the appropriate normal curve.
- To simplify the tabulation of these areas, we standardize each value of \( x \) by expressing it as a \( z \)-score, the number of standard deviations \( \sigma \) it lies from the mean \( \mu \).

\[ z = \frac{x - \mu}{\sigma} \]

\[ \text{APPLET} \]

The Standard Normal (\( z \)) Distribution

- Mean = 0; Standard deviation = 1
- When \( x = \mu \), \( z = 0 \)
- Symmetric about \( z = 0 \)
- Values of \( z \) to the left of center are negative
- Values of \( z \) to the right of center are positive
- Total area under the curve is 1.

\[ \text{APPLET} \]

Using Table 3

The four digit probability in a particular row and column of Table 3 gives the area under the \( z \) curve to the left that particular value of \( z \).

\[ \text{Area for } z = 1.36 \]

Example

Use Table 3 to calculate these probabilities:

\[ P(z \leq 1.36) = .9131 \]
\[ P(z > 1.36) = 1 - .9131 = .0869 \]
\[ P(-1.20 \leq z \leq 1.36) = .9131 - .1151 = .7980 \]

\[ \text{APPLET} \]

Using Table 3

- To find an area to the left of a \( z \)-value, find the area directly from the table.
- To find an area to the right of a \( z \)-value, find the area in Table 3 and subtract from 1.
- To find the area between two \( z \)-values, find the two areas in Table 3 and subtract one from the other.
- Remember the Empirical Rule: Approximately 95% of the measurements lie within 2 standard deviations of the mean.
1. Look for the four digit area closest to .2500 in Table 3.
2. What row and column does this value correspond to?
3. $z = -0.67$
4. What percentile does this value represent? 25th percentile, or 1st quartile (Q1)

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Working Backwards

Find the value of $z$ that has area .05 to its right.
1. The area to its left will be $1 - .05 = .95$
2. Look for the four digit area closest to .9500 in Table 3.
3. Since the value .9500 is halfway between .9495 and .9505, we choose $z$ halfway between 1.64 and 1.65.
4. $z = 1.645$

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Finding Probabilities for the General Normal Random Variable

To find an area for a normal random variable $x$ with mean $\mu$ and standard deviation $\sigma$, standardize or rescale the interval in terms of $z$.

Example: $x$ has a normal distribution with $\mu = 5$ and $\sigma = 2$. Find $P(x > 7)$.

$P(x > 7) = P(z > \frac{-7 - 5}{2})$

$= P(z > 1) = 1 - .8413 = .1587$

Example

The weights of packages of ground beef are normally distributed with mean 1 pound and standard deviation .10. What is the probability that a randomly selected package weighs between 0.80 and 0.85 pounds?

$P(0.80 < x < 0.85) = .0668 - .0228 = .0440$

Example

What is the weight of a package such that only 1% of all packages exceed this weight?

$P(x > ?) = .01$

$P(z > \frac{? - 1}{.1}) = .01$

From Table 3, $\frac{? - 1}{.1} = 2.33$

$? = 2.33(.1) + 1 = 1.233$

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The Normal Approximation to the Binomial

- We can calculate binomial probabilities using
  - The binomial formula
  - The cumulative binomial tables
  - Java applets
- When $n$ is large, and $p$ is not too close to zero or one, areas under the normal curve with mean $np$ and variance $npq$ can be used to approximate binomial probabilities.
Approximating the Binomial
✓ Make sure to include the entire rectangle for the values of x in the interval of interest. This is called the continuity correction.
✓ Standardize the values of x using
\[ z = \frac{x - np}{\sqrt{npq}} \]
✓ Make sure that np and nq are both greater than 5 to avoid inaccurate approximations!

Example
Suppose x is a binomial random variable with n = 30 and p = .4. Using the normal approximation to find P(x ≤ 10).

\[ n = 30 \quad p = .4 \quad q = .6 \]
\[ np = 12 \quad nq = 18 \]

Calculate
\[ \mu = np = 30(.4) = 12 \]
\[ \sigma = \sqrt{npq} = \sqrt{30(.4)(.6)} = 2.683 \]

Key Concepts
I. Continuous Probability Distributions
1. Continuous random variables
2. Probability distributions or probability density functions
   a. Curves are smooth.
   b. The area under the curve between a and b represents the probability that x falls between a and b.
   c. P(x = a) = 0 for continuous random variables.

II. The Normal Probability Distribution
1. Symmetric about its mean μ.
2. Shape determined by its standard deviation σ.

III. The Standard Normal Distribution
1. The normal random variable z has mean 0 and standard deviation 1.
2. Any normal random variable x can be transformed to a standard normal random variable using
\[ z = \frac{x - \mu}{\sigma} \]
3. Convert necessary values of x to z.
4. Use Table 3 in Appendix I to compute standard normal probabilities.
5. Several important z-values have tail areas as follows:

<table>
<thead>
<tr>
<th>z-Value</th>
<th>.005</th>
<th>.01</th>
<th>.025</th>
<th>.05</th>
<th>.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.58</td>
<td>2.33</td>
<td>1.96</td>
<td>1.645</td>
<td>1.28</td>
<td></td>
</tr>
</tbody>
</table>