What is Probability?

- In Chapters 2 and 3, we used graphs and numerical measures to describe data sets which were usually **samples**.
- We measured “how often” using 
  \[ \text{Relative frequency} = \frac{f}{n} \]
- As \( n \) gets larger,

```
Sample  \quad \text{And “How often”} \quad \text{Population}  \quad \text{= Relative frequency} \quad \text{Probability}
```

Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment**: Record an age
- **Experiment**: Toss a die
- **Experiment**: Record an opinion (yes, no)
- **Experiment**: Toss two coins

Basic Concepts

- A **simple event** is the outcome that is observed on a single repetition of the experiment.
  - The basic element to which probability is applied.
  - One and only one simple event can occur when the experiment is performed.
- A **simple event** is denoted by \( E \) with a subscript.
Example

• The die toss:

• Simple events: Sample space:

\[ S = \{E_1, E_2, E_3, E_4, E_5, E_6\} \]

\[ E_1 \quad \bullet \quad E_3 \quad \bullet \quad E_5 \]

\[ E_2 \quad \bullet \quad E_4 \quad \bullet \quad E_6 \]

\[ S \]

Basic Concepts

• An event is a collection of one or more simple events.

• The die toss:

– A: an odd number
– B: a number > 2

\[ A = \{E_1, E_3, E_5\} \]

\[ B = \{E_3, E_4, E_5, E_6\} \]

Basic Concepts

• Two events are mutually exclusive if, when one event occurs, the other cannot, and vice versa.

• Experiment: Toss a die

– A: observe an odd number
– B: observe a number greater than 2
– C: observe a 6
– D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

The Probability of an Event

• The probability of an event A measures “how often” we think A will occur. We write \( P(A) \).

• Suppose that an experiment is performed \( n \) times. The relative frequency for an event A is

\[ \frac{\text{Number of times A occurs}}{n} = \frac{f}{n} \]

\[ \lim_{n \to \infty} \left( \frac{f}{n} \right) \]

• If we let \( n \) get infinitely large,

\[ P(A) = \lim_{n \to \infty} \left( \frac{f}{n} \right) \]

The Probability of an Event

• P(A) must be between 0 and 1.

– If event A can never occur, \( P(A) = 0 \).

– If event A always occurs when the experiment is performed, \( P(A) = 1 \).

• The sum of the probabilities for all simple events in S equals 1.

• The probability of an event A is found by adding the probabilities of all the simple events contained in A.

Finding Probabilities

• Probabilities can be found using

– Estimates from empirical studies

– Common sense estimates based on equally likely events.

• Examples:

– Toss a fair coin. \( P(\text{Heads}) = \frac{1}{2} \)

– 10% of the U.S. population has red hair.

Select a person at random. \( P(\text{Red hair}) = .10 \)
Example

- Toss a fair coin twice. What is the probability of observing at least one head?

<table>
<thead>
<tr>
<th>1st Coin</th>
<th>2nd Coin</th>
<th>Event</th>
<th>$P(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
<td>HH</td>
<td>1/4</td>
</tr>
<tr>
<td>H</td>
<td>T</td>
<td>HT</td>
<td>1/4</td>
</tr>
<tr>
<td>T</td>
<td>H</td>
<td>TH</td>
<td>1/4</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>TT</td>
<td>1/4</td>
</tr>
</tbody>
</table>

$P(\text{at least 1 head}) = P(E_1) + P(E_2) + P(E_3) = 1/4 + 1/4 + 1/4 = 3/4$

Example

- A bowl contains three M&Ms®, one red, one blue and one green. A child selects two M&Ms at random. What is the probability that at least one is red?

<table>
<thead>
<tr>
<th>1st M&amp;M</th>
<th>2nd M&amp;M</th>
<th>Event</th>
<th>$P(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>B</td>
<td>RB</td>
<td>1/6</td>
</tr>
<tr>
<td>R</td>
<td>G</td>
<td>RG</td>
<td>1/6</td>
</tr>
<tr>
<td>B</td>
<td>R</td>
<td>BR</td>
<td>1/6</td>
</tr>
<tr>
<td>B</td>
<td>G</td>
<td>BG</td>
<td>1/6</td>
</tr>
<tr>
<td>G</td>
<td>R</td>
<td>GR</td>
<td>1/6</td>
</tr>
<tr>
<td>G</td>
<td>B</td>
<td>GB</td>
<td>1/6</td>
</tr>
<tr>
<td>G</td>
<td>G</td>
<td>GG</td>
<td>1/6</td>
</tr>
</tbody>
</table>

$P(\text{at least 1 red}) = P(RB) + P(BR) + P(RG) + P(GR) + P(GG) = 4/6 = 2/3$

Counting Rules

- If the simple events in an experiment are equally likely, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

- You can use counting rules to find $n_A$ and $N$.

Examples

Example: Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Example: Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

Example: Two M&Ms are drawn from a dish containing two red and two blue candies. The total number of simple events is:

$$4 \times 3 = 12$$
Examples

Example: A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

\[ P_5^5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120 \]

Example

- A box contains six M&Ms®, four red and two green. A child selects two M&Ms at random. What is the probability that exactly one is red?

The order of the choice is not important!

\[ C_4^1 \times C_2^1 = \frac{4!}{3!1!} \times \frac{2!}{1!1!} = 4 \times 2 = 8 \text{ ways to choose 1 red and 1 green M&M.} \]

\[ P(\text{exactly one red}) = \frac{8}{15} \]

Event Relations

- The complement of an event A consists of all outcomes of the experiment that do not result in event A. We write \( A^c \).
Example

Select a student from the classroom and record his/her hair color and gender.
– A: student has brown hair
– B: student is female
– C: student is male

• What is the relationship between events B and C?
  • A\(\cap\)C: Student does not have brown hair
  • B\(\cap\)C: Student is both male and female
  • B\(\cup\)C: Student is either male and female = all students = S

Calculating Probabilities for Unions and Complements

There are special rules that will allow you to calculate probabilities for composite events.

The Additive Rule for Unions:
For any two events, A and B, the probability of their union, \(P(A \cup B)\), is

\[P(A \cup B) = P(A) + P(B) - P(A \cap B)\]

Example: Additive Rule

Example: Suppose that there were 120 students in the classroom, and that they could be classified as follows:

<table>
<thead>
<tr>
<th></th>
<th>Brown</th>
<th>Not Brown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Female</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

\[P(A \cup B) = P(A) + P(B) - P(A \cap B)\]

= \(\frac{50}{120} + \frac{60}{120} - \frac{30}{120}\)

= \(\frac{80}{120} = \frac{2}{3}\)

Check: \(P(A \cup B) = \frac{(20 + 30 + 30)}{120}\)

A Special Case

When two events A and B are mutually exclusive, \(P(A \cap B) = 0\) and \(P(A \cup B) = P(A) + P(B)\).

A: male with brown hair
\(P(A) = \frac{20}{120}\)
B: female with brown hair
\(P(B) = \frac{30}{120}\)

\[P(A \cup B) = P(A) + P(B)\]

= \(\frac{20}{120} + \frac{30}{120}\)

= \(\frac{50}{120}\)

A and B are mutually exclusive, so that

Calculating Probabilities for Complements

We know that for any event A:

– \(P(A \cap A^c) = 0\)

Since either A or \(A^c\) must occur,

\[P(A \cup A^c) = 1\]

so that

\[P(A \cup A^c) = P(A) + P(A^c) = 1\]

\[P(A^c) = 1 - P(A)\]

Example

Select a student at random from the classroom. Define:

A: male
\(P(A) = \frac{60}{120}\)
B: female

\[P(B) = 1 - P(A)\]

= \(1 - \frac{60}{120} = \frac{40}{120}\)
Calculating Probabilities for Intersections

- In the previous example, we found \( P(A \cap B) \) directly from the table. Sometimes this is impractical or impossible. The rule for calculating \( P(A \cap B) \) depends on the idea of independent and dependent events.

Two events, \( A \) and \( B \), are said to be independent if and only if the probability that event \( A \) occurs does not change, depending on whether or not event \( B \) has occurred.

Example 1

- Toss a fair coin twice. Define
  - \( A \): head on second toss
  - \( B \): head on first toss

<p>| | |</p>
<table>
<thead>
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</tr>
<tr>
<td>TT</td>
<td>1/4</td>
</tr>
</tbody>
</table>

\[
P(A|B) = \frac{1}{2} \quad P(A|\text{not } B) = \frac{1}{2}
\]

\( P(A) \) does not change, whether \( B \) happens or not...

A and \( B \) are independent!

Example 2

- A bowl contains five M&Ms®, two red and three blue. Randomly select two candies, and define
  - \( A \): second candy is red.
  - \( B \): first candy is blue.

\[
P(A|B) = P(\text{2nd red}|1\text{st blue}) = \frac{2}{4} = \frac{1}{2}
\]

\[
P(A|\text{not } B) = P(\text{2nd red}|1\text{st red}) = \frac{1}{4}
\]

\( P(A) \) does change, depending on whether \( B \) happens or not...

A and \( B \) are dependent!

Defining Independence

- We can redefine independence in terms of conditional probabilities:

  Two events \( A \) and \( B \) are independent if and only if

  \[
P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)
  \]

  Otherwise, they are dependent.

- Once you’ve decided whether or not two events are independent, you can use the following rule to calculate their intersection.

The Multiplicative Rule for Intersections

- For any two events, \( A \) and \( B \), the probability that both \( A \) and \( B \) occur is

  \[
P(A \cap B) = P(A) \cdot P(B \text{ given that } A \text{ occurred}) = P(A)P(B|A)
  \]

- If the events \( A \) and \( B \) are independent, then the probability that both \( A \) and \( B \) occur is

  \[
P(A \cap B) = P(A) \cdot P(B)
  \]
Example 1
In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk? Define H: high risk N: not high risk

\[
P(\text{exactly one high risk}) = P(HNN) + P(NHN) + P(NNH) \\
= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243
\]

Example 2
Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?
Define H: high risk F: female

From the example, \( P(F) = .49 \) and \( P(H|F) = .08 \).
Use the Multiplicative Rule:

\[
P(\text{high risk female}) = P(F)P(H|F) = .49(.08) = .0392
\]

The Law of Total Probability
• Let \( S_1, S_2, S_3, ..., S_k \) be mutually exclusive and exhaustive events (that is, one and only one must happen). Then the probability of another event \( A \) can be written as

\[
P(A) = P(A \cap S_1) + P(A \cap S_2) + \ldots + P(A \cap S_k) \\
= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \ldots + P(S_k)P(A|S_k)
\]

Bayes’ Rule
• Let \( S_1, S_2, S_3, ..., S_k \) be mutually exclusive and exhaustive events with prior probabilities \( P(S_1), P(S_2), \ldots, P(S_k) \). If an event \( A \) occurs, the posterior probability of \( S_i \), given that \( A \) occurred is

\[
P(S_i | A) = \frac{P(S_i)P(A | S_i)}{\sum P(S_i)P(A | S_i)} \quad \text{for} \quad i = 1, 2, ..., k
\]

Example
From a previous example, we know that 49% of the population are female. Of the female patients, 8% are high risk for heart attack, while 12% of the male patients are high risk. A single person is selected at random and found to be high risk. What is the probability that it is a male?
Define H: high risk F: female M: male

\[
\begin{align*}
\text{We know:} & \\
P(F) & = .49 \\
P(M) & = .51 \\
P(H|F) & = .08 \\
P(H|M) & = .12 \\
P(H|F \cap M) & = .09 \\
P(H|F \cap M) & = .09 \\
\end{align*}
\]

\[
P(M|H) = \frac{P(M)P(H|M)}{P(M)P(H|M) + P(F)P(H|F)} \\
= \frac{.51(.12)}{.51(.12) + .49(.09)} \\
= \frac{.061}{.5113 + .4409} \\
= .061
\]
Random Variables

• A quantitative variable \( x \) is a random variable if the value that it assumes, corresponding to the outcome of an experiment is a chance or random event.
• Random variables can be discrete or continuous.
• Examples:
  - \( x \) = SAT score for a randomly selected student
  - \( x \) = number of people in a room at a randomly selected time of day
  - \( x \) = number on the upper face of a randomly tossed die

Probability Distributions for Discrete Random Variables

• The probability distribution for a discrete random variable \( x \) resembles the relative frequency distributions we constructed in Chapter 1. It is a graph, table or formula that gives the possible values of \( x \) and the probability \( p(x) \) associated with each value.

We must have 
\[ 0 \leq p(x) \leq 1 \] \[ \sum p(x) = 1 \]

Example

• Toss a fair coin three times and define \( x \) = number of heads.

Probability Distributions

• Probability distributions can be used to describe the population, just as we described samples in Chapter 1.
  - Shape: Symmetric, skewed, mound-shaped…
  - Outliers: unusual or unlikely measurements
  - Center and spread: mean and standard deviation. A population mean is called \( \mu \) and a population standard deviation is called \( \sigma \).

The Mean and Standard Deviation

• Let \( x \) be a discrete random variable with probability distribution \( p(x) \). Then the mean, variance and standard deviation of \( x \) are given as

\[
\text{Mean} : \mu = \sum xp(x) \\
\text{Variance} : \sigma^2 = \sum (x - \mu)^2 p(x) \\
\text{Standard deviation} : \sigma = \sqrt{\sigma^2}
\]
Example

- The probability distribution for $x$ the number of heads in tossing 3 fair coins.

- Shape?
  - Symmetric; mound-shaped

- Outliers?
  - None

- Center?
  - $\mu = 1.5$

- Spread?
  - $\sigma = .688$

Key Concepts

I. Experiments and the Sample Space
1. Experiments, events, mutually exclusive events, simple events
2. The sample space
3. Venn diagrams, tree diagrams, probability tables

II. Probabilities
1. Relative frequency definition of probability
2. Properties of probabilities
   a. Each probability lies between 0 and 1.
   b. Sum of all simple-event probabilities equals 1.
3. $P(A)$, the sum of the probabilities for all simple events in $A$

III. Counting Rules
1. $mn$ Rule; extended $mn$ Rule
2. Permutations: $P(n,r) = \frac{n!}{(n-r)!}$
3. Combinations: $C(n,r) = \frac{n!}{r!(n-r)!}$

IV. Event Relations
1. Unions and intersections
2. Events
   a. Disjoint or mutually exclusive: $P(A \cap B) = 0$
   b. Complementary: $P(A) = 1 - P(A')$

V. Discrete Random Variables and Probability Distributions
1. Random variables, discrete and continuous
2. Properties of probability distributions
   $0 \leq p(x) \leq 1$ and $\sum p(x) = 1$
3. Mean or expected value of a discrete random variable
   $\mu = \sum xp(x)$
4. Variance and standard deviation of a discrete random variable
   $\sigma^2 = \sum(x - \mu)^2 p(x)$
   $\sigma = \sqrt{\sigma^2}$